DOCUMENT RESUME

ED 141 115 SE 022 614

TITLE	Mathematics 7-8 Handbook, 1976 Reprint.
INSTITUTION	New York State Education Dept., Albany. Bureau of General Education Curriculum Development.
PUB DATE	76
NOTE	210p.; For 1973 Edition, see ED 079 115; Contains occasional light type
EDRS PRICE	MF-$0.83 HC-$11.37 Plus Postage.
DESCRIPTORS	Curriculum; *Curriculum Guides; Grade 7; Grade 8; *Instruction; *Learning Activities; Mathematical Enrichment; Mathematics Education; Secondary Education; *Secondary School Mathematics; *Teaching Guides

ABSTRACT

This handbook, prepared for teachers of grades 7 and 8, provides suggestions for teaching various aspects of the mathematics courses outlined in the syllabus of the New York State Education Department. The handbook deals with twelve units: sets, systems of numeration, natural numbers, whole numbers, positive rationals, integers, the complete set of rationals, reals, ratio and related topics, geometry, statistics, and probability. For each unit there is a review of important concepts, a brief discussion of student needs in the area, suggestions for learning activities, and descriptions of suggested enrichment activities. (SD)

****
* Documents acquired by ERIC include many informal unpublished *
* materials not available from other sources. ERIC makes every effort *
* to obtain the best copy available. Nevertheless, items of marginal *
* reproducibility are often encountered and this affects the quality *
* of the microfiche and hardcopy reproductions ERIC makes available *
* via the ERIC Document Reproduction Service (EDRS). EDRS is not *
* responsible for the quality of the original document. Reproductions *
* supplied by EDRS are the best that can be made from the original. *
****

U S DEPARTMENT OF HEALTH
EDUCATION & WELFARE
NATIONAL INSTITUTE OF
EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL NATIONAL INSTITUTE OF EDUCATION POSITION OR POLICY

ED141115

MATHEMATICS
7-8
HANDBOOK

BEST COPY AVAILABLE

SE 022 614

The University of the State of New York
THE STATE EDUCATION DEPARTMENT
Bureau of General Education Curriculum Development
Albany, New York 12234

1784

MATHEMATICS

7 - 8

HANDBOOK

1976 Reprint

The University of the State of New York
THE STATE EDUCATION DEPARTMENT
Bureau of General Education Curriculum Development
Albany, New York 12234

THE UNIVERSITY OF THE STATE OF NEW YORK

Regents of The University (with years when terms expire)

1981 Theodore M. Black, A.B., Litt.D., LL.D., Pd.D., D.C.L., L.H.D., Chancellor - Sands Point
1987 Carl H. Pforzheimer, Jr. A.B., M.B.A., D.C.S., H.H.D., Vice Chancellor- Purchase
1978 Alexander J. Allan, Jr., LL.D., Litt.D.- - - - - - - - - - Troy
1981 Joseph C. Indelicato, M.D., L.H.D. - - - - - - - - - - - - Brooklyn
1986 Kenneth B. Clark, A.B., M.S., Ph.D., LL.D., L.H.D., D.Sc.- Hastings on Hudson
1983 Harold E. Newcomb, B.A.- - - - - - - - - - - - - - - - - - - Owego
1988 Willard A. Genrich, LL.B., L.H.D., LL.D. - - - - - - - - - Buffalo
1982 Emlyn I. Griffith, A.B., J.D.- - - - - - - - - - - - - - - - Rome
1977 Genevieve S. Klein, B.S., M.A. - - - - - - - - - - - - - - Bayside
1981 William Jovanovich, A.B., LL.D., Litt.D., L.H.D. - - - - Briarcliff Manor
1983 Mary Alice Kendall, B.S. - - - - - - - - - - - - - - - - - - Irondequoit
1984 Jorge L. Batista, B.A., J.D. - - - - - - - - - - - - - - - - Bronx
1982 Louis E. Yavner, LL.B. - - - - - - - - - - - - - - - - - - - New York
1979 Laura B. Chodos, B.A., M.A.- - - - - - - - - - - - - - - - - Clifton Park
1980 Martin C. Barell, B.A., I.A., LL.B.- - - - - - - - - - - - Great Neck

President of The University and Commissioner of Education
Ewald B. Nyquist

Executive Deputy Commissioner of Education
Gordon M. Ambach

Deputy Commissioner for Elementary, Secondary, and Continuing Education
Thomas D. Sheldon

Associate Commissioner for Instructional Services
William L. Bitner

Assistant Commissioner for General Education and Curricular Services
Vivienne Anderson

Director, Division for Curriculum Services
Gordon E. Van Hooft

Chief, Bureau of General Education Curriculum Development
Herbert Bothamley

Director, Division of General Education
Ted T. Grenda

Chief, Bureau of Mathematics Education
Frederic Paul

CONTENTS

Page

Foreword.......... vii

Unit I. SETS.......... 1
- Symbols.......... 1
- Identification of Sets.......... 2
- The Empty Set.......... 4
- Finite and Infinite Sets.......... 5
- Equal Sets and Equivalent Sets.......... 6
- Subsets.......... 7
- Union and Intersection of Sets.......... 10
- Venn Diagrams.......... 12

Unit II. SYSTEMS OF NUMERATION.......... 17
- Tallying.......... 17
- Egyptian Numerals.......... 18
- Roman Numerals.......... 18
- The Hindu-Arabic or Decimal System.......... 19
- Place Value in the Decimal System.......... 20
- Writing a Numeral as a Polynomial and in Positional Notation.......... 21
- Writing, Reading, and Spelling Numerals in Decimal Notation.......... 22
- Factors, Exponents.......... 23
- Digits in Various Bases.......... 26
- Place Value Tables.......... 26
- Translating From Any Base To Any Other Base.......... 30
- Addition in Various Bases.......... 31
- Subtraction in Various Bases.......... 32
- Multiplication in Various Bases.......... 33
- Additional Enrichment Activities.......... 34

Unit III. THE SET OF NATURAL NUMBERS.......... 39
- Properties of the Natural Numbers.......... 39

Unit IV. THE SET OF WHOLE NUMBERS.......... 44
- The Number Line.......... 48
- Mathematical Sentences.......... 50
- Divisibility.......... 52
- Primes.......... 54
- Order of Operations.......... 55
- Magic Squares.......... 56
- Figurate Numbers.......... 58
- Missing Digits.......... 60
- Alphametics.......... 60

Page

Unit V. THE SET OF POSITIVE RATIONALS.......... 63
Subsets of the Fractions.......... 64
Simplifying Fractions.......... 65
Increasing the Terms of a Fraction.......... 67
Operations With Fractions.......... 68
Properties of Positive Rationals.......... 76

Unit VI. THE SET OF INTEGERS.......... 84
Addition.......... 85
Subtraction.......... 86
Multiplication.......... 88
Division.......... 91
Absolute Value.......... 92

Unit VII. THE COMPLETE SET OF RATIONALS.......... 93
Addition of Rationals.......... 94
Subtraction of Rationals.......... 95
Multiplication of Rationals.......... 95
Division of Rationals.......... 96
Negative Exponents.......... 97
Scientific Notation.......... 98
Measurement.......... 98
Significant Digits.......... 101
Square Root.......... 102

Unit VIII. THE SET OF REAL NUMBERS.......... 105

Unit IX. RATIO, PROPORTION, PER CENT, VARIATION.......... 108
Ratio.......... 108
Proportion.......... 109
Per Cent.......... 110
Variation.......... 117

Unit X. GEOMETRY.......... 120
Undefined Terms.......... 120
Point.......... 121
Line.......... 121
Plane and Lines in a Plane.......... 123
Angle.......... 124
Measurement of Angles.......... 126
Angles as Applied to Lines and Planes.......... 126
Polygon.......... 128
Triangle.......... 129
Quadrilaterals.......... 131
Circle.......... 132
Measurement and Constructions.......... 134
Measurement of Distances.......... 140
Measurement of Areas.......... 141
Proportions as Applied to Similar Triangles.......... 145
Polyhedrons.......... 149

Page

Prisms.. 152
Pyramids...................................... 154
Cones... 156
Cylinders..................................... 156
Spheres....................................... 158
Surface Areas of Solids....................... 159
Volumes of Solids............................. 160
Space Visualization........................... 161
Coordinate Geometry Using the Number Line.......... 161
Coordinate Geometry Using the Number Plane......... 162
Graphing Open Sentences....................... 166
Graphing - Equalities......................... 167
Graphing - Inequalities....................... 168
Glossary of Terms for Geometry................ 171

Unit XI. STATISTICS........................... 182
Tabulation of Data............................ 183
Representation of Data........................ 187
Interpretation of Graphs...................... 190
Measures of Central Tendency.................. 191
Data and Data Collection...................... 193

Unit XII. PROBABILITY......................... 195

FOREWORD

This handbook has been prepared in order to provide teachers of seventh year and eighth year mathematics with suggestions for teaching various aspects of the course. It is intended that it be used with and supplement the Department syllabus, *Mathematics Courses for the Seventh Year and Eighth Year.*

A committee of teachers and Department personnel met in May 1966 to develop an outline for seventh and eighth grade mathematics. This committee designed a recommended fundamental course, the scope and organization of which reflected several guiding principles: (1) the framework must be consistent and reasonable, (2) the content should be based on the 7X and 8X materials as modified by evaluations and recommendations resulting from their use, and (3) the course must provide good articulation and continuity with both *Mathematics K - 6, A Proposed Program* and *Ninth Year Mathematics, Course I - Algebra*, as revised in 1965.

The outline, *Mathematics Courses for the Seventh Year and Eighth Year*, which this committee constructed, contains no optional topics. For suggestions in this area reference was made in the syllabus to 7X and 8X materials, some of which are now out of print. In order to provide teachers of seventh and eighth grade mathematics with a background of optional materials, and also to flesh out the "what" of the outline with some suggestions concerning the "how," this handbook was developed.

The suggestions should prove particularly helpful to beginning teachers of mathematics. Such teachers should realize, however, that the methods and illustrations used are not prescribed procedures; teachers will want to experiment with alternate approaches as these come to their attention.

Nor should teachers feel that everything in this handbook must be "covered" in every class. A number of areas which are designated as optional have been included in this development. Stronger pupils may be led to explore some of these optional topics, but it is probable that not all of these topics can be touched upon by any one group. It is the intent of this handbook to make such explorations profitable when opportunities for this activity arise. The optional topics open windows upon fascinating vistas, but this additional material should not be introduced in classes where this is done at the expense of adequate treatment of the required topics.

Experienced teachers, too, will welcome new suggestions for presenting old and new topics. Such background supplementation will make available to them a wider variety of approaches and techniques to vary their presentations. Some of the suggestions will provide the teacher with procedures which will enable greater emphasis to be placed on certain aspects of "modern" mathematics.

The original draft of the manuscript was written by Mrs. Irna De Long, mathematics teacher, Albany Public Schools. She was advised to draw freely on the 7X and 8X publications of the Department. Fredric Paul, associate, Bureau of Mathematics Education, worked closely with Mrs. De Long, and checked the manuscript as it was being written.

Aaron Buchman, associate, Bureau of Mathematics Education, under temporary assignment to the Bureau of Secondary Curriculum Development, organized, edited, and prepared the final manuscript for publication.

Herbert Bothamley, *Chief*
Bureau of General Education
Curriculum Development

Gordon E. Van Hooft, *Director*
Division for Curriculum Services

I. SETS

The topic of sets will need thorough development in some classes, but review will be sufficient in other classes depending on how much has been taught previously.

The term *set* is undefined but can be considered to be a collection of objects or concepts which are called *elements* or *members* of the set. The symbol epsilon, $\in$, is used to indicate that a certain object is an element of a certain set. "$3 \in A$" is read "3 is an element of set A". "$5 \notin A$" is read, "5 is not an element of set A". Sets may be identified by *listing* in braces the elements of a set such as {0,1,2,3,4}. The same set may be *described in words* as {all whole numbers less than or equal to four}. Also, some sets may be *graphed* on a number line. It is suggested that simple representation of sets on the number line be taught at this time.

The largest set under consideration is the *universal set*; its symbol is usually U. A *finite* set contains a definite number of elements; such a set is {a,b,c,d}. An *infinite* set contains an endless number of elements. For example, {1,2,3,4,...} is an infinite set; the three dots denote the endless pattern. The *null* set contains no members. The set of all living dogs with wings is an example of the null set, which is symbolized by { } or ϕ. *Equal* sets contain the same elements, not necessarily in the same order. *Equivalent* sets are sets with the same cardinal number, so that a one-to-one correspondence exists. Sets which contain no members in common are *disjoint* sets.

When elements of the universal set are grouped in various ways to form many sets, these sets are called *subsets*. The null set is a subset of every set. Every set is a subset of itself. All subsets are *proper subsets* except the set itself. If set A is a subset of B, this can be indicated by the symbol, $A \subseteq B$. If set A is a proper subset of B, this can be indicated by the symbol, $A \subset B$.

The *union* of set A and set B is the set consisting of all the elements in both sets; no element may be listed more than once. The union of set A and set B may be indicated by the use of the symbols $A \cup B$. The *intersection* of set A and set B is the set containing all the elements common to both sets. The intersection of set A and set B may be indicated by the use of the symbols $A \cap B$.

SYMBOLS

$\in$ is an element of
$\notin$ is not an element of
{ } or ϕ null set, empty set
$\subset$ proper subset

$\subseteq$ subset
$\cap$ intersection, "and"
$\cup$ union, "or"
U universal set

Identification of Sets

Concept: Meaning of a set.

(1) Any collection or group of objects or concepts which have some property or characteristic in common may be called a set. The chairs in your classroom make up a set. The properties they have in common are that they are chairs and that they are in your classroom.

Look around the classroom and identify collections or groups of objects that can be classified as sets.

Answer:
The set of all chairs in the room
The set of all desks in the room
The set of all books in the room
The set of all boys in the room
The set of all girls in the room
The set of all teachers in the room
The set of all lights in the room
The set of all clocks in the room
The set of all tables in the room

Note: The sets listed are a representative sample of sets which may be named. Pupils may need direction in choosing sets containing only one element.

(2) Sets do not have to be composed of objects. They may be collections or groups of concepts or ideas such as numbers, points, planes, and lines.

Name some sets which are not composed of physical objects.

Answer:
The set of all even numbers
The set of all odd numbers
The set of all natural numbers
The set of all rational numbers
The set of all numbers greater than X (where X is some given number)
The set of all numbers greater than X but less than Y
The set of all points on a given line
The set of all line segments which form a given geometric figure
The set of all planes parallel to a given plane

Concept: Symbols for a set.

(3) The members of a set are called its elements. A set may be of any size and may contain no elements, one element, any finite number of elements, or an infinite number of elements. A set

is referred to by listing the names of the elements in the set within braces or by describing the elements in the set. One particular set may be referred to as "the set of all natural numbers greater than 4 and less than 10." This same set may be referred to as {5, 6, 7, 8, 9}.

Below are sets which are identified by describing the elements in the set. Identify these same sets by listing their elements within braces.

(a) The set of all even numbers between 9 and 21
(b) The set of all odd numbers between 10 and 20
(c) The set of all the months in the year which begin with the letter F
(d) The set of all the states in the United States completely surrounded by water
(e) The set of all men who served as President of the United States at any time during the period 1935 - 1962
(f) The set of all states in the United States which border on the Atlantic Ocean
(g) The set of all planets in our solar system whose orbits are inside the orbit of Jupiter

Answers: (a) {10, 12, 14, 16, 18, 20}
(b) {11, 13, 15, 17, 19}
(c) {February}
(d) {Hawaii}
(e) {Roosevelt, Truman, Eisenhower, Kennedy}
(f) {Florida, Georgia, South Carolina, North Carolina, Virginia, Maryland, Delaware, New Jersey, New York, Connecticut, Rhode Island, Massachusetts, New Hampshire, Maine}
(g) {Mercury, Venus, Earth, Mars}

(4) *Below are sets which are identified by listing all the elements in each set. Identify each of these sets by describing the elements in the set. There may be several ways of describing each set of elements.*

(a) {2, 4, 6, 8, 10}
(b) {Alaska, Hawaii, California, Oregon, Washington}
(c) {April, June, November, September}
(d) {21, 22, 23, 24, 25}
(e) {4, 8, 12, 16, 20, 24}

Answers: (a) The set of all natural numbers between, but not including, 1 and 11
(b) The set of all states in the United States that border on the Pacific Ocean
(c) The set of all months in the year that have 30 days, and only 30 days
(d) The set of all natural numbers greater than 20 and less than 26
(e) The set of all natural numbers less than 25 that are divisible by 4

Concept: The use of dots in listing elements in a set.

(5) Three dots may be used in listing the elements in a set which contains a large number of elements and such that the elements form a definite pattern. In the set $\{1, 2, 3, 4, \ldots, 586\}$, the three dots mean "continuing in the same manner up to and including the number 586." In the set $\{1, 2, 3, 4, \ldots\}$ the three dots mean "continuing in the same manner without end."

Identify each of the following sets by listing the elements in each set, using the three dots wherever they are useful.

(a) The set of all even natural numbers
(b) The set of all odd natural numbers up to and including 3339
(c) The set of all letters in the alphabet
(d) The set of all states in the United States
(e) The set of all fractions between 1 and 2 whose denominator is 13

Answers: (a) $\{2, 4, 6, 8, \ldots\}$
(b) $\{1, 3, 5, 7, \ldots, 3339\}$
(c) {A, B, C, D, ..., Z}
(d) {Alabama, Alaska, Arizona, Arkansas, ..., Wyoming}
(e) $\{\frac{13}{13}, \frac{14}{13}, \frac{15}{13}, \frac{16}{13}, \ldots, \frac{26}{13}\}$

The Empty Set

Concept: Meaning of and symbol for the empty set.

(1) Sometime a set will contain no elements at all. Such a set is called the empty or null set. The symbol for the null set is ϕ or { }. The set of all two-headed elephants in your classroom is the null set. The set of numbers greater than 2 and less than 1 is the null set. Care must be taken not to confuse the number 0 with the null set. Zero is a number. It may be listed as an element of a set. The null set contains *no* element. The set of numbers which can be substituted for x to make $x + 2 = 2$ a true statement is $\{0\}$. The set of numbers which can be substituted for x to make $x + 2 = x$ a true statement is ϕ or { }. This is a very common point of confusion with pupils, and some time should be devoted to preventing and clearing up any possible confusion.

List the elements in each of the following sets.

(a) The set of all live dinosaurs in your school
(b) The set of all numbers greater than 5 and less than 3
(c) The set of all numbers that may be substituted for y to make the equation $6 - y = 6$ a true statement

(d) The set of all numbers that may be substituted for x to make the equation $\frac{9}{x} = 0$ a true statement

(e) The set of all numbers that may be substituted for x to make the equation $\frac{x}{9} = 0$ a true statement

Answers: (a) ϕ, (b) ϕ, (c) $\{0\}$, (d) ϕ, (e) $\{0\}$,

Finite and Infinite Sets

Concept: Definition of finite and infinite sets.

(1) A finite set is a set which contains a certain number of elements. This number may be fantastically huge, but as long as it is definitely a number, then the set is a finite set. The number of molecules of water in the Pacific Ocean is a very large number. The number is constantly changing, and there is no way of determining the number, but at any given instant there is a definite number of molecules of water present, and it is possible to name many large finite numbers which exceed this number. The set of all molecules of water in the Pacific Ocean is therefore a finite set. An infinite set is a set which contains an endless number of elements. The set of all the natural numbers is an infinite set.

It is impossible to determine whether certain sets pertaining to objects in the universe are finite or infinite because we do not know if the universe is finite or infinite. When pupils ask if the set of all electrons in the universe is finite or infinite, the best answer is that we do not know.

The pupils should be given experience at determining the difference between sets that have a very large number of elements and an infinite set.

Identify each of the following sets as being finite or infinite.

(a) The set of all the grains of sand on a given beach
(b) The set of all even numbers greater than 10
(c) The set of all fractions between the numbers 8 and 9
(d) The set of all points on a given line
(e) The set of all points on a line segment 0.0008 of an inch in length
(f) The set of all people now alive on earth
(g) The set of all human beings who are or who ever were alive on earth
(h) The set of all protons in our solar system
(i) $\{1, 2, 3, 4, \ldots 99999\}$

(j) {1, 2, 3, 4, ...}

Answers:
(a) Finite
(b) Infinite
(c) Infinite
(d) Infinite
(e) Infinite
(f) Finite
(g) Finite
(h) Finite
(i) Finite
(j) Infinite

Equal Sets and Equivalent Sets

Concept: Definition of equal sets.

A set may be described in more than one way. The set of all odd numbers greater than 2 and less than 8 is the set {3, 5, 7}. The set of all prime numbers greater than 2 and less than 8 is the set {3, 5, 7}. Each of these sets contains the same elements. They are equal sets. Two sets are equal if, and only if, they contain the same elements. The order of the elements within the set does not matter.

Concept: Definition of equivalent sets

Equivalent sets are sets which contain the same *number* of elements. When two sets contain the same number of elements, there exists a one-to-one correspondence between the two sets. For every element in the first set, there is a corresponding element in the second set and for every element in the second set there is a corresponding element in the first set. Such sets are also called matching sets. The sets {2,4,6,8} and {Mary, Tom, Joe, Pete} are equivalent or matching sets. The sets {1,2,3,4,...} and {2,4,6,8,...} are also matching sets.

Equal sets are always equivalent, but equivalent sets are not necessarily equal. The set of all odd numbers greater than one and less than 9 and the set of all prime numbers greater than 2 and less than 8 are equal sets {3,5,7} = {3,5,7}, but they are also equivalent since the cardinal number of each set is three.

The set of the primary colors, {red, blue, yellow}, is equivalent to the set {Mary, Jane, Sue}, but they are not equal.

In the blanks below, write the letter of the set in column B which is equal to the set after the blank in column A.

Column A	*Column B*
_____(1) {17, 15, 9, 11}	(A) {9, 11, 15, 17}
_____(2) The set of all natural numbers greater than 20 and less than 10	(B) {11, 17, 23, 13, 19}

Column A

______(3) The set of all two-digit numbers
______(4) $\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$
______(5) The set of all prime numbers between 10 and 24
______(6) The set of prime factors of 21
______(7) The set of all digits in the number 193
______(8) The set of all natural numbers divisible by 2
______(9) $\{\frac{3}{1}, \frac{12}{3}, \frac{25}{5}\}$
______(10) {A, B, C, D, ...,H}

Column B

(C) {7, 3}
(D) {2, 4, 6, 8, ...}
(E) ϕ
(F) {H, G, D, E, ...,A}
(G) {3, 4, 5}
(H) {10, 11, 12, ...,99}
(I) {1, 3, 9}
(J) {0.25, 0.5, 0.75}

Answers: (1) A, (2) E, (3) H, (4) J, (5) B (6) C, (7) I, (8) D, (9) G, (10) F

In the blanks below, write the letter of the set in column B which is equivalent to the set after the blank in column A.

Column A

______(1) {1, 2, 3}
______(2) {A, B, C....H}
______(3) $\{\frac{2}{3}, \frac{3}{4}, \frac{5}{8}, \frac{7}{9}\}$
______(4) the set of wheels on a bicycle

Column B

(A) the set of legs on a normal dog
(B) {3,5,7,13,45,97,26,60}
(C) {9, 10}
(D) {A, B, C}

Answers: (1) D, (2) B, (3) A, (4) C

Subsets

Concept: Meaning of universal set and subsets.

The largest set under consideration is called the universal set. Its symbol is *U*. Elements within the universal set may be grouped in various ways to form many sets. These new sets are called subsets of the universal set.

If every element in set A is also an element in set B, then set A is a subset of set B. If the universal set is the set of all pupils in the classroom, then the set of all boys is a subset of the universal set. The set of all girls in the classroom is another subset of the universal set. The set of all boys with dark hair is a subset of the set of boys in the classroom. Every set is considered a subset of itself. Also, the null set is considered a subset of every set. The total number of subsets in a set consisting of one element is therefore 2. A set consisting of two elements will have 4 possible subsets. A set consisting of 3 elements will have 8 subsets. An examination of this pattern shows that the total number of possible subsets of a set that contains n elements is 2^n. The total number of subsets of a set of 5 elements is 2^5 or 32 subsets.

When we say that set A is a subset of set B, it is possible that set A and set B each have the same elements. If set A is a subset of set B, and if set B contains at least one element which is not in set A, then we can call set A a proper subset of set B.

If set A is a subset of set B, this can be indicated by the symbol, $A \subseteq B$. If set A is a proper subset of set B, this can be indicated by the symbol, $A \subset B$.

In the examples below, the universal set is the set of all the whole numbers. In each example, indicate whether or not one set is a subset of the other set and identify which set is the subset.

(a) A = {8, 9, 10} B = {2, 4, 6, 8, 9, 10}
(b) A = {0, 6} B = {6, 0}
(c) A = {1, 3, 5, 7, 9} B = {1, 2, 3, ...,10}
(d) A = {4, 6, 9} B = {1, 2, 3}
(e) A = {5, 7, 9} B = {7, 5, 9}

Answers:
(a) Set A is a subset of set B
(b) Set A is a subset of set B and set B is a subset of set A
(c) Set A is a subset of set B
(d) Neither set is a subset of the other set
(e) Set A is a subset of set B and set B is a subset of set A

(2) *Is every set a subset of itself?*

Answer: Yes. Every element in set A is also an element in set A. Therefore, set A is a subset of set A.

(3) *If two sets are equal, is each set a subset of the other set?*

Answer: Yes. Each element in one set is also an element in the other set. Therefore, each set is a subset of the other set.

(4) The letter *U* is used to indicate the universal set. The capital letters such as A and B are used to indicate subsets of the universal set. The universal set and several subsets A, B, C, and D are as follows.

$U = \{0, 1, 2, 3, \ldots, 10\}$
$A = \{1, 4, 7, 10\}$
$B = \{0, 2, 4, 6, 8, 10\}$
$C = \{1, 2, 3, \ldots, 10\}$
$D = \{2, 6\}$

Indicate which of the following statements are true and which of them are false.

(a) Set A is a subset of set B
(b) Set A is a subset of set C
(c) Set B is a subset of set A
(d) Set A is a subset of set D
(e) Set C is a subset of set D
(f) Set B is a subset of set D
(g) Set C is a subset of set A
(h) Set D is a subset of set B
(i) Set B is a subset of set C
(j) Set D is a subset of set A

Answers:

(a) False (c) False (e) False (g) False (i) False
(b) True (d) False (f) False (h) True (j) False

Concept: The number of subsets in a set.

(5) *If the null set is a subset of every set, list all the possible subsets in the set* {0}.
Answer: {0}, ϕ

(6) *List all the possible subsets of the set* {a, b}.
Answer: {a}, {b}, {a, b}, ϕ

(7) *List all the possible subsets of the set* {4, 5, 6}.
Answer: {4}, {5}, {6}
{4, 5}, {4, 6}, {5, 6}
{4, 5, 6}
ϕ

(8) *List all the possible subsets in the set* {1, 2, 3, 4}.
Answer: {1}, {2}, {3}, {4}
{1,2}, {1,3}, {1,4},
{2,3}, {2,4}, {3,4},
{1,2,3}, {2,3,4}, {1,3,4}, {1,2,4}
{1,2,3,4}
ϕ

(9) *How many subsets are there in a set containing only one element? two elements? three elements? four elements? five elements? n elements?*
Answers: 2, 4, 8, 16, 32, 2^n

Union and Intersection of Sets

Concept: The union of two sets.

(1) Consider the set $A = \{1, 5, 9, 14, 17\}$ and the set $B = \{3, 7, 14, 15\}$. A third set C may be obtained by grouping the elements in set A with the elements in set B without repeating elements which are in both sets. The new set would be $C = \{1,3,5,7,9,14,15,17\}$. Set C is the union of sets A and B. The union of two sets, A and B, is the set composed of every element in set A plus every element in set B which is not in set A. The union, of course, includes every element which is in set A, every element which is in set B, and every element which is in both set A and set B.

The union of set A and set B may be indicated by the use of the symbols $A \cup B$. $A \cup B$ is read "A or B" or "A union B".

Indicate by listing, the union of each pair of the following sets.

(a) $A = \{1, 6, 8\}$ and $B = \{5, 9, 11\}$.
(b) $A = \{a, b, c, d\}$ and $B = \{a, b, k\}$.
(c) $A = \{a, 5, b, 7\}$ and $B = \{7, b, 5, a\}$.
(d) A = the set of all boys in your class.
B = the set of all brown-eyed pupils in your class.
(e) A = the set of all odd natural numbers.
B = the set of all natural numbers.
(f) $A = \{1, 6, 9\}$ and $B = \phi$.
(g) A = {John, Peter, Mary} and B = {California}

Answers:
(a) $C = \{1,5,6,8,9,11\}$
(b) $C = \{a,b,c,d,k\}$
(c) $C = \{a,b,5,7\}$
(d) C = the set defined by the list consisting of all the boys plus all the brown-eyed girls in the class.
(e) $C = \{1,2,3, \ldots\}$
(f) $C = \{1,6,9\}$
(g) C = {John, California, Mary, Peter}

Instead of using set C to indicate the union of A and B the answer to a) could be written $A \cup B = \{1,5,6,8,9,11\}$ and similarly for the remaining answers.

Concept: The intersection of two sets.

(2) Consider the sets $A = \{1,2,3,4,5,6,7,8\}$ and $B = \{2,4,6,8,10,12\}$. The set $C = \{2,4,6,8\}$ is composed of every element that is an element in both set A and set B. Such a set is called the intersection of sets A and B. The intersection of two sets, A and B, is the set of all elements

which are elements common to both set A and set B. The intersection of set A and set B may be indicated by use of the symbols A ∩ B. A ∩ B is read "A and B" or "A intersection B".

Given sets A, B, C, D, and E below, indicate the intersections requested.

A = {1,3,9,11}
B = {1,3,5,9,13}
C = {3,9,15,17,21}
D = {9,17,21,25}
E = ϕ

(a) Indicate the intersection of sets A and B.
(b) Indicate the intersection of sets B and D.
(c) Indicate the intersection of sets C and A.
(d) Indicate the intersection of sets C and E.
(e) Indicate the intersection of sets A, B, and C.

Answers:

a) C = {1,3,9}
b) C = {9}
c) C = {3,9}
d) C = ϕ
e) C = {3,9}

Instead of using set C to indicate the intersection of A and B, the answer to a) could be written A ∩ B = {1,3,9} and similarly for the remaining answers.

Concept: Meaning of disjoint sets.

(3) If two sets have no elements in common, they are said to be disjoint sets. The intersection of disjoint sets is the null set.

In each of the following, indicate whether or not sets A and B are disjoint.

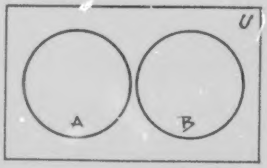
(a) A = {1,3,4,5} and B = {2,4,6,8,10}.
(b) A = the set of natural numbers less than 10.
B = the set of natural numbers greater than 9.
(c) A = the set of all even natural numbers.
B = the set of all odd natural numbers.
(d) A = the set of all numbers greater than 5 and less than 8.
B = the set of all numbers less than 6.
(e) A = the set of all even numbers.
B = {0}.

Answers:
(a) No, they are not disjoint.
(b) Yes, they are disjoint.
(c) Yes, they are disjoint.
(d) No, they are not disjoint.
(e) No, they are not disjoint.

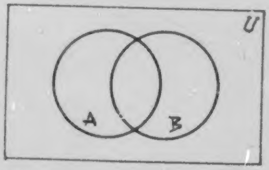
Venn Diagrams

Sets may be represented pictorially by the use of a rectangle to represent the universal set with circles to represent subsets of the universal sets. There are five possible ways of representing pictorially a universal set and two subsets of the universal set.

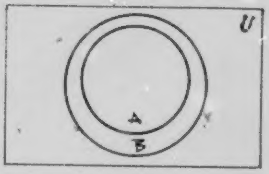
(1) Set A and set B are disjoint sets.



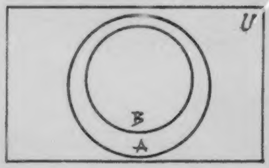
(2) Set A and set B have an intersection but are not proper subsets of each other.



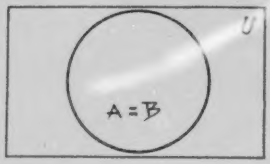
(3) Set A is a proper subset of set B.



(4) Set B is a proper subset of set A.



(5) Set A and set B are equal sets. They are subsets of each other.



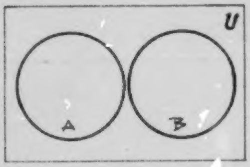


Figure I

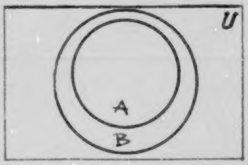


Figure II

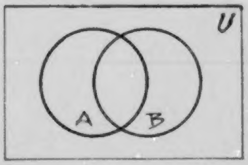


Figure III

Ask pupils to perform the following exercises.

(a) In Figure I, shade in the area that represents the union of sets A and B.

(b) In Figure II, shade in the area that represents the intersection of sets A and B.

(c) In Figure III, shade in the area that represents the union of sets A and B.

(d) In Figure III, shade in the area that represents the intersection of sets A and B.

(e) In Figure II, shade in the area that represents the union of sets A and B.

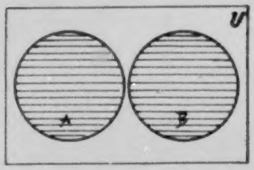
(f) In Figure I, shade in the area that represents the intersection of sets A and B.

(g) In Figure I, shade in the area that represents the intersection of set A and the universal set.

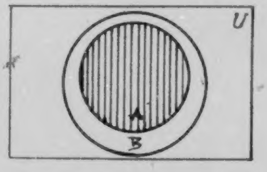
(h) In Figure I, shade in the area that represents the union of set B and the universal set.

Answers:

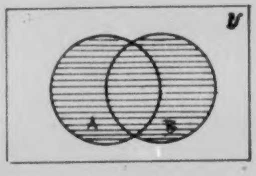
(a)



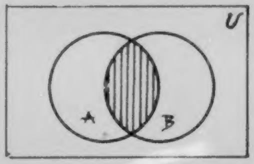
(b)



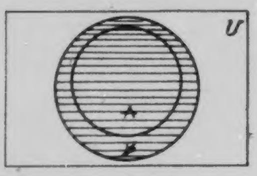
(c)



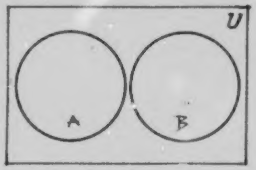
(d)



(e)

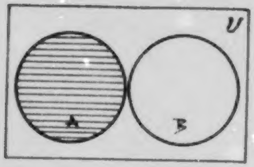


(f)

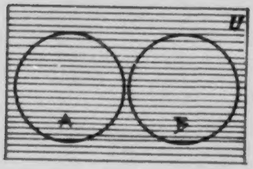


(No intersection)

(g)



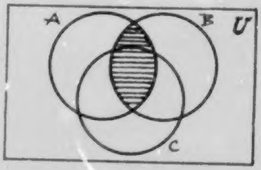
(h)



Below are pictorial representations of a universal set and three subsets of the universal set, with several unions and intersections shaded in.

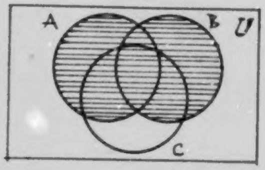
The intersection of set A with set B.

A ∩ B.



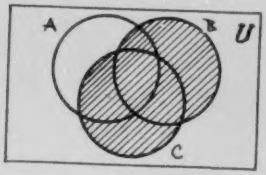
The union of set A with set B.

A ∪ B.



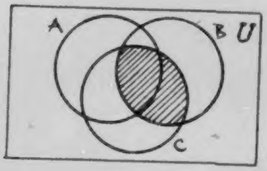
The union of set B with set C.

B ∪ C.

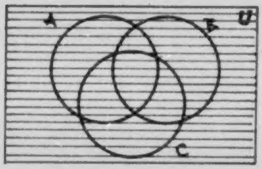


The intersection of set B and set C.

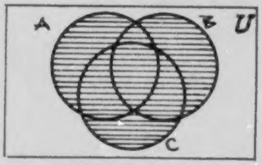
B ∩ C.



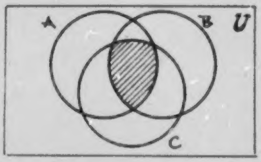
The union of set A (or any set) with the universal set
$A \cup U$.



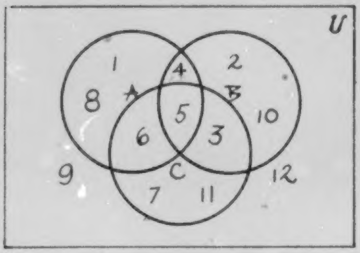
The union of set C and the union of sets A and B.
$(A \cup B) \cup C$.



The intersection of set C with the intersection of sets A and B.
$(A \cap B) \cap C$.



Concept: Pictorial representation of three mutually intersecting sets.



At the left is a Venn diagram representing the universal set U and subsets A, B, and C. List the elements (in this case, the numerals) found within the unions and intersections listed below.

(a) $A \cup B$
(b) $A \cup C$
(c) $B \cap C$
(d) $A \cap C$
(e) $C \cup B$

(f) $A \cup B \cup C$
(g) $B \cup U$
(h) $A \cap B$
(i) $A \cap U$
(j) $A \cap B \cap C$

Answers:

(a) {1,2,3,4,5,6,8,10}
(b) {1,3,4,5,6,7,8,11}
(c) {3,5}
(d) {5,6}
(e) {2,3,4,5,6,7,10,11}
(f) {1,2,3,4,5,6,7,8,10,11}
(g) {1,2,3 ...12}
(h) {4,5}
(i) {1,4,5,6,8}
(j) {5}

II. SYSTEMS OF NUMERATION

Concept: A system of numeration is a system used to name numbers. Systems of numeration developed as a result of the necessity for counting. Students may enjoy *short* discussions on the Egyptian, Babylonian, and Roman systems of numeration before studying the Hindu-Arabic system. Students should be aware that other systems of numeration existed before the decimal system, but too great an emphasis should not be placed on these systems.

At this level, students should be taught to distinguish between *number*, an idea or concept which is independent of language, and *numeral*, or symbol used to name a number. We could write any of the following symbols to represent this number of stones -- o o o o:

4, IV, 1111, four, $\frac{8}{2}$, 10-6, 16 x $\frac{1}{4}$, quatro, 3 + $\frac{3}{3}$, , (6 + -2)

One-to-one-correspondence exists if exactly one thing or object is matched with exactly one other thing or object. Four letters and four numbers are in a one-to-one correspondence as follows:

$$\begin{array}{cccc} a & b & c & d \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 2 & 3 & 4 & 5 \end{array}$$

It is easy to find examples of sets that can be put into a one-to-one correspondence; for example, chairs — desks, floor — room, flag — classroom and star — state. There is also the possibility of a one-to-many correspondence and a many-to-one correspondence; for example, teacher to students and desks to classroom, respectively.

Tallying

As man's possessions increased he needed some way to keep track of them. He did this by tallying — he made a mark, or a notch on a stick, for each item he had. He might have used ||||| ||||| ||| (13) to represent thirteen objects.

A short introduction to the Egyptian and Roman systems would provide some historical background to show that our decimal system was but another step in the development of numeration. The elementary grades have provided some work on the ancient number systems. Charts of the symbols and their values may be made. No written work should be assigned, but some pupils might enjoy doing some as enrichment.

Egyptian Numerals

The Egyptians grouped the tally symbols so that each ten would be represented as 𓎆. This, of course, immediately made their naming of numbers simpler. They used the following symbols to name their numbers:

1	𓎆	𓍢	𓆼	𓂭
1	10	100	1,000	10,000
vertical staff	heel-bone	coiled rope	lotus flower	bent finger

The Egyptian numerals could be repeated and were additive. So that to represent forty-two thousand, there hundred seventy-six, the Egyptians would write:

𓏺𓏺𓏺𓏺𓏺𓏺 𓎆𓎆𓎆𓎆𓎆𓎆𓎆 𓍢𓍢𓍢 𓆼𓆼 𓂭𓂭𓂭𓂭

Evaluating from right to left yields:

𓂭𓂭𓂭𓂭 = 10,000 + 10,000 + 10,000 + 10,000 = 40,000
𓆼𓆼 = 1,000 + 1,000 = 2,000
𓍢𓍢𓍢 = 100 + 100 + 100 = 300
𓎆𓎆𓎆𓎆𓎆𓎆𓎆 = 10 + 10 + 10 + 10 + 10 + 10 + 10 = 70
𓏺𓏺𓏺𓏺𓏺𓏺 = 1 + 1 + 1 + 1 + 1 + 1 = 6
42,376

Roman Numerals

The Roman numeral system is another system of numeration The Roman system used these symbols:

I	V	X	L	C	D	M
1	5	10	50	100	500	1000

This system is repetitive, additive, subtractive, and multiplicative as shown in the following examples.

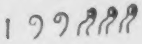
XXX = 10 + 10 + 10 = 30 (show repetition and addition)

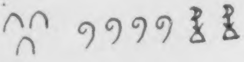
IV = 5 - 1 = 4 (shows subtraction)

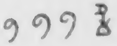
$\overline{V}$ = 5,000 (a line over a numeral multiplies its value by 1000)

Exercises which may be used as enrichment.

Write in Hindu-Arabic numerals:

(a) 

(b) 

(c) 

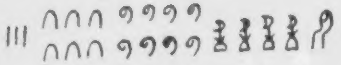
Answers: (a) 30,301 (b) 2,435

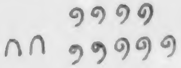
(c) 1,300

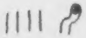
Write in Egyptian numerals:

(a) 14,863 (b) 920 (c) 10,004

Answers:

(a) 

(b) 

(c) 

Write in Hindu-Arabic numerals:

(a) XXV (b) MDCCLII (c) MCDXXVI (d) XLIV (e) $\overline{IV}$

Write in Roman numerals:

(a) 2,009 (b) 468 (c) 7,304 (d) 10,000

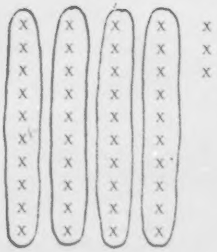
The Hindu-Arabic or Decimal System

In the systems of numeration described above new symbols had to be introduced for larger numbers. The beauty of our Hindu-Arabic system is that we use only ten symbols or digits to write any number — 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Therefore, we call this the decimal system of numeration because the word decimal comes from the Latin *decem*, meaning ten. The fact that we are able to write any number using only the ten digits is accounted for by two things: (1) the zero, used as a placeholder (was introduced between 600-800 A.D.) and (2) the idea of place value. The digits as we know them today were not in complete use until the invention of printing.

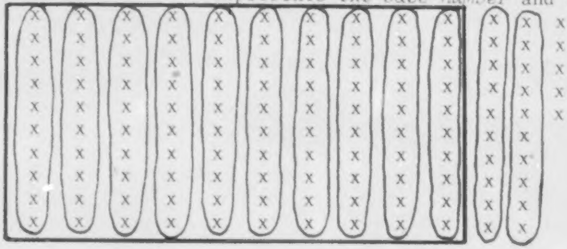
There are ten symbols probably because we have ten fingers on our hands. The decimal system is also called base ten. The terms *base*, *scale*, and *radix* all mean the same thing.

Cardinal numbers are used to tell how many; {3,4,5,6} has four elements and its cardinal number is four. *Ordinal numbers* are numbers used to name a position; four is the second number in {3,4,5,6}.

Place Value in the Decimal System



The significance of the Hindu-Arabic system lies in the concept of place value. Suppose we want to represent the number of x's at the left. We would count 4 groups of ten and 3 ones, and we would write down the numeral to represent the number of x's as 43, where $4 \times 10 + 3 \times 1 = 40 + 3 = 43$ This also shows that this system of numeration is *additive*. The numeral 10 (read one-zero) in the decimal system means one ten and zero ones or $1 \times 10 + 0 \times 1$. It is well to note that in any base, *10* represents the *base number* and that in the decimal system ten is not represented by a new digit or new symbol but only a combination of two digits already used. This is important since later on in base five, for example, the student must realize that there is no numeral 5. To count the number of x's above, we count and group by tens. We will have twelve tens and four ones, but when we have ten tens we call this one hundred. We then have:



12 (tens) + 4 (ones) = 1 (hundred) + 2 (tens) + 4 (ones)
or
$1 \times 100 + 2 \times 10 + 4 \times 1$

and we write this as 124 and read it as one hundred twenty-four (not one-hundred *and* twenty-four)

The place values in base ten would be:

...	ten thousands	thousands	hundreds	tens	ones or units	tenths	hundredths	...
...	10,000	1,000	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	...
...	$10\cdot10\cdot10\cdot10$	$10\cdot10\cdot10$	$10\cdot10$	10	1	$\frac{1}{10}$	$\frac{1}{10\cdot10}$	...

The places to the right of the one's place are what we commonly call the decimal places, although all the places are decimal places because they indicate powers of ten; it is not sufficient to say multiples of ten. It is important that we are aware of the pattern in the table involving the factors of ten. The three dots to the left and right mean that the table is continued indefinitely.

Writing a Numeral as a Polynomial and in Positional Notation

Concept: Writing a numeral as a polynomial is the same as writing it as an expanded numeral.

Example 1: Write 385 as a polynomial or as an expanded numeral.

Answer: $\boxed{3 \times 100 + 8 \times 10 + 5 \times 1}$

The same numeral written in *positional notation* is just 385.

Example 2: Write 1469 as an expanded numeral.

positional notation	expanded form
1469	= 1 x 1000 + 4 x 100 + 6 x 10 + 9 x 1

Writing, Reading, and Spelling Numerals in Decimal Notation

Concept: In forming the numerals we usually write the digits in groups of three (periods), separated by commas for convenience in reading. This numeral

541,467,328,913

billions millions thousands ones

would be read as five hundred forty-one billion, four hundred sixty-seven million, three hundred twenty-eight thousand, nine hundred thirteen. (Notice: the writing of our numerals in groups of three thus makes them in a certain sense based on one thousand.)

Numeral	Prefix	Meaning of Prefix	Other familiar words containing the stem or prefix
billion	bi-	2	bicycle, bicuspid, biceps, bimonthly
trillion	tri-	3	trio, triplets, triple
quadrillion	quadr-	4	quadrangle, quadrilateral, quadruped
quintillion	quin-	5	quintet, quintuplets, quinary
sextillion	sex- or hex-	6	sextet
septillion	sept- or hept-	7	September, originally 7th month before Julius' July and Augustus' August
octillion	oct-	8	October, octave
nonillion	non-	9	nonagon
decillion	dec-	10	decimal, decade
undecillion duodecillion			See Webster's New Collegiate Dictionary

As mentioned before the word *and* is not used at all in reading a whole number, but is reserved for reading the decimal point in a numeral such as 329.47, which is read three hundred twenty-nine *and* forty-seven hundredths. The numerals between multiples of ten, except for one through twenty, are hyphenated, as in forty-five and ninety-two; three hundred is not hyphenated. Since numbers expressed in billions and trillions are seen in the newspaper almost every day, we should be familiar with the preceding table of number names (which would make a good Latin lesson).

Can anyone read the numeral 18,446,744,073,709,551,615?

Students are amused by seeing such a long numeral. It may also be expressed as $2^{64} - 1$. It is read eighteen quintillion, four hundred forty-six quadrillion, seven hundred forty-four trillion, seventy-three billion, seven hundred nine million, five hundred fifty-one thousand, six hundred fifteen.

Factors, Exponents.

Exponents are introduced at this point to facilitate the writing of a numeral in expanded form as well as their use later on in scientific notation.

Concept: Factors of a product are numbers which, when multiplied, give the product. In this unit the factors will be members of the set of natural numbers. In the example 2 x 3 = 6, 2 and 3 are factors of 6 because when we multiply 2 by 3, the product is 6. In this unit a *factor of a given number* may be said to be a natural number which will divide that given number without a remainder; one and the given number itself are always factors of the number. In the example above, 2 is a factor of 6 because 6 ÷ 2 = 3, a natural number; 3 is also a factor of 6; 4 is not a factor of 6 because 6 ÷ 4 will not give a whole number as the quotient. In the example $2\cdot3\cdot3\cdot5 = 90$. (the dot here means the same thing as the earlier multiplication sign, "x"), 2 is a factor of 90, 3 is a factor of 90, 3 again is a factor of 90 (3 is a factor twice), and 5 is a factor of 90. Any product containing one or more of the factors 2, 3, 3, or 5, used no more times than shown here, will also be a factor of 90. In other words, if $2\cdot3\cdot3\cdot5 = 90$, a list of all the factors of 90 would be 1, 2, 3, 5, $2\cdot3$, or 6, $3\cdot3$ or 9, $2\cdot5$ or 10, $3\cdot5$ or 15, $2\cdot3\cdot3$ or 18, $2\cdot3\cdot5$ or 30, $3\cdot3\cdot5$ or 45, and $2\cdot3\cdot3\cdot5$ or 90. When a factor is repeated as $5\cdot5\cdot5$, we can write this as 5^3. This means that 5 is used as a factor 3 times (*not* that 5 is multiplied by itself 3 times), and is equal to 125. In the two examples below express the powers in factor form and as a product.

Example 1: $2^4 = \boxed{2\cdot2\cdot2\cdot2 = 16}$ (*not* $2\cdot4 = 8$).

Example 2: $10^3 = \boxed{10\cdot 10\cdot 10 = 1000}$

In 10^3, the ^{3} is written smaller than the 10, and is written above and to the right of it. The ^{3} is called an *exponent*, the 10 is the *base*, 10^3 or 1000 is a power of ten (in this case the third power of ten), and 10^3 is read 10 to the third power.

Concept: An exponent indicates how many times we use the base as a factor (*not* how many times we multiply the base by itself). Exceptions are the first power which is always equal in value to the base: i.e. $10^1 = 10$; $5^1 = 5$; and the zero power which is always equal in value to one: i.e. $10^0 = 1$; $5^0 = 1$.

Further illustrations of powers are given in the following table: (Note that usually in *typewritten* copy, the exponent is only rolled up and not printed smaller as it should be.)

Number	Exponent	Base	Read as	Meaning	Equals
10^4	4	10	ten to the fourth power	$10\cdot 10\cdot 10\cdot 10$	10,000
6^2	2	6	6 square(d) or 6 to the second power	$6\cdot 6$	36
2^3	3	2	two cube(d) or two to the third power	$2\cdot 2\cdot 2$	8
3^5	5	3	three to the fifth power	$3\cdot 3\cdot 3\cdot 3\cdot 3$	243
8^1	1	8	eight to the first power	8	8
5^0	0	5	five to the zero power	1	1

The values assigned to the first and zero powers of a base can be made "reasonable" by developing a pattern, though this is not a "proof".

Example 1: If $5^3 = 125$, then $5^4 = \boxed{5x5x5x5 = (5x5x5)\ x5 = 125 \times 5 = 625}$

Example 2: If $5^6 = 15{,}625$, $5^5 = \boxed{15{,}625 \div 5 = 3{,}125}$

Example 3: If $5^2 = 25$, $5^1 = \boxed{25 \div 5 = 5}$

Example 4: If $5^1 = 5$, $5^0 = \boxed{5 \div 5 = 1}$

Concept: If the exponent is increased by 1, the new value is equal to the previous value multiplied by the base. If the exponent is decreased by 1, the new value is equal to the previous value divided by the base.

Example 5: If $10^2 = 100$, $10^1 = \boxed{100 \div 10 = 10}$

Example 6: If $10^1 = 10$, $10^0 = \boxed{10 \div 10 = 1}$

A number then may be written in three main ways:

1. *Positional notation:* 274
2. *Polynomial or expanded form:* $2 \times 100 + 7 \times 10 + 4 \times 1$
3. *Exponential form:* $2 \times 10^2 + 7 \times 10^1 + 4 \times 10^0$.

Concept: Scientific notation is a method of writing numbers as a product of a number between one and ten and a power of ten.

1. $30 = 3 \times N$

   $\frac{30}{3} = N = 10 = 10^1$

   $\therefore 30 = 3 \times 10^1$

2. $400 = 4 \times N$

   $\frac{400}{4} = N = 100 = 10^2$

   $\therefore 400 = 4 \times 10^2$

3. $5000 = 5 \times N$

   $\frac{5000}{5} = N = 1000 = 10^3$

   $\therefore 5000 = 5 \times 10^3$

After practise, students will begin to verbalize the counting off rule, let them discover it - do not feed it to them!

Work in bases other than ten should be introduced only to strengthen an understanding of the decimal system for the less able students, and as *enrichment* for the more capable. Excessive time spent doing operations in other bases would be better used in practise in our own system of numeration. Purely as enrichment material - addition, subtraction and multiplication in other bases are shown. Basic work on systems other than base ten is now a part of most elementary school mathematics programs, therefore a minimum of reteaching should be necessary.

Digits in Various Bases

Concept: The number of digits used in any one of the various bases is the same as the base number. For example, base 2 uses 2 digits, (0 and 1). The digits used are:

base 2 (binary system): 0, 1

base 5 (quinary system): 0, 1, 2, 3, 4

base 8 (octal system): 0, 1, 2, 3, 4, 5, 6, 7

base 10 (decimal system): 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

base 12 (duodecimal system): 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, t, e

Place Value Tables

Place value tables for some of the whole number places in various bases would look like this.

Base ten

positional notation	...	10,000	1,000	100	10	1	...
factors	...	10x10x10x10	10x10x10	10x10	10	1	...
exponents	...	10^4	10^3	10^2	10^1	10^0	...

Base 8 place values as equivalent numerals in base ten

...	4096	512	64	8	1	...
...	8•8•8•8	8•8•8	8•8	8	1	...
...	8^4	8^3	8^2	8^1	8^0	...

Base 5 place values as equivalent numerals in base ten

...	625	125	25	5	1	...
...	5•5•5•5	5•5•5	5•5	5	1	...
...	5^4	5^3	5^2	5^1	5^0	...

Base 2 place values as equivalent numerals in base ten

...	16	8	4	2	1	...
...	2•2•2•2	2•2•2	2•2	2	1	...
...	2^4	2^3	2^2	2^1	2^0	...

Concept: To change a numeral in any other base to a base ten numeral. It is important for students to remember that place values are dependent on the base. i.e.

$$
\begin{aligned}
234_{\text{six}} &= 2 \times 6^2 + 3 \times 6^1 + 4 \times 6^0 = \\
&2 \times 36 + 3 \times 6 + 4 \times 1 = \\
&72 + 18 + 4 = 94
\end{aligned}
$$

Concept: To change a numeral from base ten to any other base.
There are three methods which may be used to convert 47_{10}, for example, into a numeral in base 5.

Example 1:

Draw a picture of this problem.

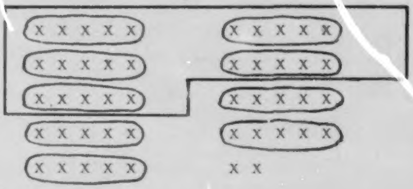
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x

Now group these 47 x's in base 5 (which we have done before). We group them in 5's, then 25's, if necessary, when grouping larger amounts.

1x25 + 4x5 + 2x1

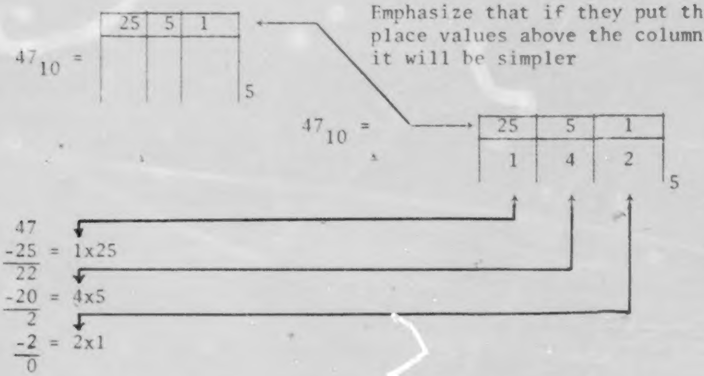
142_5

$\therefore 47_{10} = 142_5$



Method 2:

Since we know the place values in base 5 are ... 125, 25, 5, 1, we can ask, *how many 125's are in 47_{10}?* None. *How many 25's?* One. Therefore we will have a 1 in the 25's place.


47_{10} =
25
5
1
5
47_{10} =
25
5
1
1
4
2
5

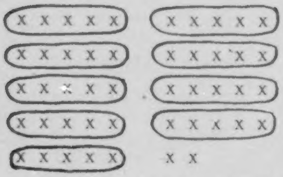
Emphasize that if they put the place values above the columns it will be simpler

$$
\begin{array}{r l}
47 & \\
-25 & = 1x25 \\
\hline
22 & \\
-20 & = 4x5 \\
\hline
2 & \\
-2 & = 2x1 \\
\hline
0 &
\end{array}
$$

Subtracting 25 from 47 we get 22. We find no groups of 25 in 22. *What can we get?* Fives. *How many groups of five in 22?* Four. Therefore we'll have a 4 in the five's place, and 4x5 = 20. Subtracting 20 from 22, we get a difference of 2. There are no fives in 2, but we have 2 ones; the 2 then goes in the one's place, Subtracting 2 from 2 we get zero and our problem is finished.

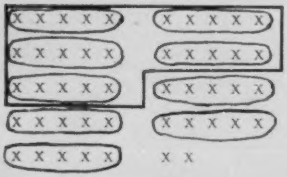
$$\therefore 47_{10} = 142_5$$

Method 3:



Divide $\begin{array}{r} 9 \text{ r } 2 \\ 5\overline{)47} \end{array}$

What does this "quotient" 9 and remainder 2 mean? This "quotient" 9 shows there are 9 groups of 5 and 2 ones remaining. *Can we show this on the diagram?* Yes (see above, left). *How can we find the number of groups of 5-fives (or 25's) in 9 groups of five (or 45)?*

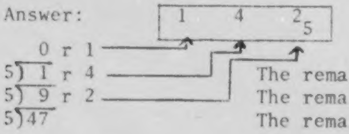


Divide $\begin{array}{r} 1 \text{ r } 4 \\ 5\overline{)9} \end{array}$

What does this "quotient 1, and remainder 4 mean? This "quotient" 1 means there is 1 group of 25 with 4 (the remainder) groups of 5 left over; we also have the 2 ones left from the first division. If we divide again by 5, what will that tell us?

$$\begin{array}{r} 0 \text{ r } 1 \\ 5\overline{)1} \end{array}$$

The question asked by this division is, *how many 5-twenty-fives (or 125's) are contained in 1 twenty-five. What does the "quotient" 0 and remainder 1 mean here?* The "quotient" 0 means there are zero groups of 125 with 1 group of 25 remaining. *Do we have to divide again?* No, for all further divisions will give quotients of zero. *What then does the repeated division by 5, as a whole, tell us?*



The remainder 1 is the number of 25's
The remainder 4 is the number of 5's
The remainder 2 is the number of 1's

In other words the remainders are the digits in the numeral we want; the *remainder in the first division is always the digit in the ones place.*

Method 3 may also be done in the following form

$$\begin{array}{r|l l} 5 & 47 & R\ 2 \\ \hline 5 & 9 & R\ 4 \\ \hline 5 & 1 & R\ 1 \\ \hline & 0 & \qquad 142_5 \end{array}$$

Where the remainders are read as shown

Translating From Any Base To Any Other Base

Concept: Change the numeral from the given base to base ten, then change the base ten numeral to the desired base.

i.e. 146_{seven} → base ten numeral → base four numeral

$146_{seven} \rightarrow 83 \rightarrow \longrightarrow 1103_{four}$

A method for converting from one base to another: without conversion to base ten could make a good enrichment topic for independent development.

Example: Change 324_7 to a base 5 numeral.

The converting is done by dividing the number written in the original base by the new base number to which the conversion is being made - the new base number must also be expressed of as a numeral in the original base

i.e. 5 in base 7 is 5, hence the divisor when changing base 7 to base 5 is 5.

8 in base 7 is 11, hence the divisor when changing base 7 to base 8 must be 11.

The division is then performed in the original base with the method consistent to that of changing from base ten to another base.

Base 7

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	15
3	0	3	6	12	15	21	24
4	0	4	11	15	22	26	33
5	0	5	13	21	26	34	42
6	0	6	15	24	33	42	51

$$\begin{array}{r} 45 \\ 5\overline{)324_7} \\ 26 \\ \hline 34 \\ 34 \\ \hline \boxed{0} \end{array}$$

Let b = base number

$\boxed{0}$ → 1st Remainder = b^0 digit

$$\begin{array}{r} 6 \\ 5\overline{)45_7} \\ 42_7 \\ \hline \boxed{3} \end{array}$$

$\boxed{3}$ → 2nd Remainder = b^1 digit

$$\begin{array}{r} 1 \\ 5\overline{)\,6} \\ \underline{5} \\ \boxed{1} \end{array} \rightarrow \text{3rd Remainder} = b^2 \text{ digit}$$

$$\begin{array}{r} 0 \\ 5\overline{)\,1} \\ \underline{0} \\ \boxed{1} \end{array} \rightarrow \text{4th Remainder} = b^3 \text{ digit.}$$

Check

$324_7 = 1130_5$ $\qquad$ $1130_5 = 165_{10}$

$324_7 = 165_{10}$

Addition In Various Bases

Example: Add in base 8:

$$\begin{array}{r} 43_8 \\ \underline{+27_8} \end{array}$$

Method I: May be done with the use of an addition table.

Base 8

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

$$\begin{array}{r} {}^{1} \\ 43_8 \\ \underline{27_8} \\ 72_8 \end{array}$$

or:

Method 2:

$$\begin{aligned} & 43_8 + 27_8 \\ = \; & 40 + 3 + 20 + 7 \\ = \; & 40 + 20 + 3 + 7 \\ = \; & 60 + 12 \\ = \; & 60 + 10 + 2 \\ = \; & 70 + 2 \\ = \; & \boxed{72_8} \end{aligned}$$

$$\begin{array}{r} 43 \\ +27 \\ \hline \end{array} \qquad \begin{array}{r} {}^{1} \\ 43 \\ +27 \\ \hline 2 \end{array}$$

Of course we do not go through all this, we think: 3 + 7 = 12, 1 eight and 2 ones; the 2 (ones) go in the one's place, the 1 (eight) is added in the eight's place (commonly called carrying). One eight plus 4 eights plus 2 eights is 7 eights; the 7 eights goes in the eights place. Our sum therefore is $\boxed{72_8}$.

Subtraction in Various Bases

Concept: The important thing in subtraction is that in "borrowing" or better "regrouping" or "exchanging", we work with whatever base we are in, not necessarily ten.

Method 1: Subtraction may be done using the addition table; if subtraction is done by the additive method. Renaming may be necessary.

i.e. $\begin{array}{r} 64_8 \\ -12_8 \\ \hline \end{array}$

Ask: What must be added to 2_8 to yield 4_8? From the table we find $2_8 + 2_8 = 4_8$. What must be added to 1_8 to yield 6_8? From the table we find $1_8 + 5_8 = 6_8$. Then: 52_8 is the difference.

$\begin{array}{r} 61_8 \\ -17_8 \\ \hline \end{array}$

Ask: What must be added to 7_8 to yield 1_8? Upon inspection of the addition table we find there is no base 8 number which may be added to 7_8 to produce the sum 1_8. Renaming is necessary.

$$\begin{array}{rcrcl} 61_8 & = & 60_8 + 1_8 & = & 50_8 + 11_8 \\ \underline{-17_8} & = & \underline{-10_8 - 7_8} & & \underline{10_8 - 7_8} \\ & & & & 40_8 + 2_8 = 42_8 \end{array}$$

Ask: What must be added to 7_8 to yield 11_8? Upon inspection of the addition table we find $2_8 + 7_8$ yields 11_8. What must be added to 10_8 to yield 50_8? Since the zeros in this case are place holders, only the 1 and 5 digits must be considered. Upon inspection of the table. $1_8 + 4_8 = 5_8$, the answer then to the problem is $40_8 + 2_8 = 42_8$.

A check may be made by adding or changing to base ten, adding and changing back to base eight.

Method 2:

Example, Subtract, base 8: $\begin{array}{r} 61 \\ -17 \\ \hline \end{array}$

Think:

61 = 6(8) + 1(1) = 5(8) + 1(8) + 1(1) = 5(8) + 10(1) + 1(1) = 5(8) + 11(1) and 17 = 1(8) + 7(1)

$$\text{Then: } \begin{array}{r} 5(8) + 11(1) \\ -1(8) - 7(1) \\ \hline 4(8) + 2(1) \end{array}$$

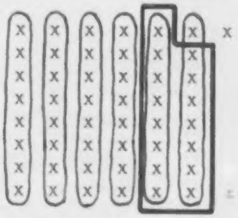
Therefore:

$$\begin{array}{r} 61 \\ \underline{-17} \end{array} \quad = \quad \begin{array}{r} 5\ 11 \\ \underline{-1\ \ 7} \end{array}$$ and since 11 - 7 = 2 in base 8,

$$\begin{array}{r} 61 \\ \underline{-17} \end{array}$$

Answer: $\boxed{42_8}$

A picture would look like this:



Multiplication In Various Bases

Method 1: Using a multiplication table.

Base 8

x	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	10	12	14	16
3	0	3	6	11	14	17	22	25
4	0	4	10	14	20	24	30	34
5	0	5	12	17	24	31	36	43
6	0	6	14	22	30	36	44	52
7	0	7	16	25	34	43	52	61

$$\begin{array}{r} 213_8 \\ x4_8 \\ \hline 1054_8 \end{array}$$

Method 2. Using the distributive law. $213_8 \times 4_8$

$$\begin{array}{rl} 213 & \\ \times\ 4 & \\ \hline 14 & = 4 \times 3 \\ 40 & = 4 \times 10 \\ \underline{1000} & = 4 \times 200 \end{array}$$

Answer: 1054_8

$$\begin{aligned} \text{or } 4(213) &= 4\ (200 + 10 + 3) \\ &= 4 \times 200 + 4 \times 10 + 4 \times 3 \\ &= 1000 + 40 + 14 \end{aligned}$$

Answer: $\boxed{1054_8}$

Sets of tables for addition and multiplication in the various bases should be constructed and used by pupils. [For the less able student in mathematics, the base ten multiplication table may be made and used in the everyday class situation. Many times a student understands how to do an example, but gets lost because of a lack of knowledge of the fundamental combinations - especially for multiplication.]

Additional Enrichment Activities:

1. *Abacus.* The following electric abacus could be made for the binary notation. The materials needed would be 7 Christmas bulbs (110 volt) and sockets and a piece of wood about one foot long. Depending on their type, the sockets could be mounted on top of the wood or underneath. In either case, the lights must be of the type which may be connected in parallel; that is, when one is turned out the others will remain lighted.

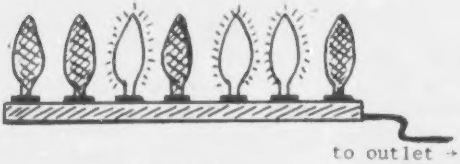


Figure 1

The lighted bult would then represent a 1, the bulb off would represent a 0; thus figure 1 would represent 0010110_2 or just 10110_2.

2. *Nim.* This ancient game for two may be played with sticks, pebbles, buttons, checkers, or any other small objects. The counters, say checkers, are arranged in three rows. The two players alternately remove checkers according to the following rules: (1) In each play one player may take away checkers from only one row. (2) He may take away as *many* checkers as he wishes but *at least one.* (3) He may take away a whole row if he wants to. The player who takes the last checker wins the game. The number of counters in each row doesn't matter. For example, let us play with 3, 5 and 7 checkers in the rows. The arrangement is

shown below in figure 2 with the binary representation of the number of counters.

o o o	1 1
o o o o o	1 0 1
o o o o o o o	1 1 1
	2 2 3

Figure 2

Summing the 1's above (in base ten) we get 2, 2, 3. This combination is odd since it contains an odd number (3). In order to win the game we must leave our opponent with an even combination (all even numbers or zeros in the sum). If we remove one counter from any row, say the middle one, the result will look like that in figure 3.

o o o	1 1
o o o o	1 0 0
o o o o o o o	1 1 1
	2 2 2

Figure 3

This leaves our opponent with an even combination (2, 2, 2) and we will win no matter what our opponent does. Let us continue the game to the finish. Our opponent takes three checkers from the bottom row, leaving us with

o o o	1 1
o o o o	1 0 0
o o o o	1 0 0
	2 1 1

If we take all the counters in the top row, we leave our opponent with an even combination.

o o o o	1 0 0
o o o o	1 0 0
	2 0 0

Our opponent takes all the checkers in the top row, leaving us with an odd combination.

o o o o	1 0 0
	1 0 0

We just take this whole row including the last checker and thus win the game. If we have an even combination our opponent must make it *odd* at his move. If the game starts with an even combination and it is our move we just have to play and hope our opponent makes a mistake (assuming he doesn't know how to win every time). If the game starts with an odd combination the first player can always win, otherwise the second player can always win. This game can also be played at the chalkboard. The teacher may want to take on the whole class before asking them how Nim is related to the binary system.

3. *The Tower of Hanoi.* This is a very interesting game or puzzle, consisting of a board with three pegs and 6 discs as illustrated in figure 4. The pegs are one-quarter inch dowels about 4 inches long, set into three holes. The 6 discs are one-half inch thick varying from a 2-inch diameter down to a three-quarter inch diameter in steps of one-quarter inch. Each disc has a five-sixteenth inch center hole.

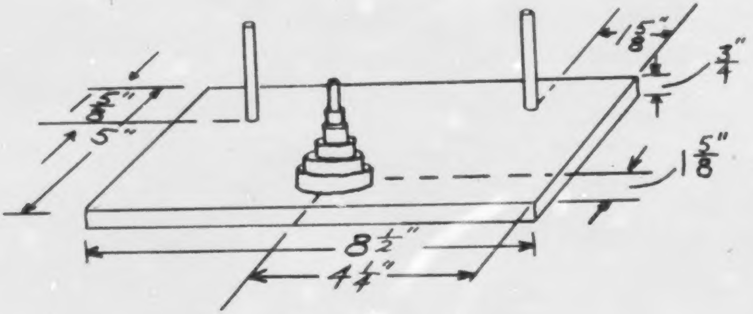


Figure 4

On one peg rests a number of the discs arranged so that the largest is on the bottom, the next smaller on top of that and the next smaller one on top of that one until the smallest one is on top. The object of the game is to transfer all the discs from one peg to one of the other two pegs by: (1) moving only one disc at a time and (2) making certain that *no disc is ever allowed to rest on one smaller than itself.*

If we start with only one disc, it takes one transfer to move that disc to another peg. If we start with two discs, it takes three transfers, the smaller disc needing two transfers, the larger disc one transfer. In fact, the largest disc will always need only one transfer, if the moving was done in the least possible number of moves. The interesting table below shows the number of discs to start, the number of transfers each disc makes, and the total number of transfers for the given number of discs.

Number of discs on peg at start	Number of transfers each disc makes (smallest, ..., largest disc)	Total Number of transfers
1	$2^0 =$	1
2	$2^1 + 2^0 =$	3
3	$2^2 + 2^1 + 2^0 =$	7
4	$2^3 + 2^2 + 2^1 + 2^0 =$	15
5	$2^4 + 2^3 + 2^2 + 2^1 + 2^0 =$	31
6	$2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 =$	63
.	. .	.
.	. .	.
.	. .	.
n	2^{n-1} + ---------------- + $2^0 =$	$2^n - 1$

From the table we can see that with four discs to start, the smallest disc moves eight (2^3) times, the next larger disc moves four (2^2) times, the next larger two (2^1) times and the largest disc moves once (2^0). The total number of transfers is $2^n - 1$ or $2^4 - 1$ or 15.

Is there a way of telling which discs to move each time? Yes, using the binary system. *Suppose we start with four discs. Number the discs one to four, from smallest to largest, then number the moves ($2^4 - 1 = 15$) from one to fifteen and place the binary numerals next to the move number* (see table 1).

Each place in a binary numeral represents a disc. The one's place represents disc number one, the smallest disc; the two's place represents disc number two, the next larger disc, and so forth. To find which disc to move, go over from right to left until a 1 is reached. If the 1 were in the third place over, we would move disc number three.

Now that we know which disc is moved, we must decide on where to move it. If there are no other digits to the left of the 1 (the disc we are going to move), as at move 8, we place the disc on the peg with no other discs on it. If there are other digits to the left of the first 1, count over to the next 1 (which represents a larger disc which has already been moved). If there is no zero or an even number of zeros between the 1's, as at move 6, the smaller disc goes on top of the larger disc, to which the next 1 refers. If there is an odd number of zeros between the ones, as at move 10, the smaller disc goes on the peg not containing the larger disc to which the next 1 refers.

Table 1 gives the instructions needed to transfer the four discs.

Table 1

1	1	Move disc 1 (smallest) onto empty peg (no digits to left of the 1)
2	10	Move disc 2 onto empty peg (no digits to left of the 1)
3	11	Move disc 1 on top of disc 2 (no zeros between the 1's)
4	100	Move disc 3 onto empty peg (no digits to left of 1)
5	101	Move disc 1, but not onto disc 3 (odd number of zeros between 1's)
6	110	Move disc 2 onto disc 3 (no digits between 1's)
7	111	Move disc 1 onto disc 2
8	1000	Move disc 4 onto empty peg
9	1001	Move disc 1 onto disc 4 (even number of zeros between 1's)
10	1010	Move disc 2, but not on disc 4 (odd number of zeros between 1's)
11	1011	Move disc 1 onto disc 2
12	1100	Move disc 3 onto disc 4
13	1101	Move disc 1, but not on disc 3
14	1110	Move disc 2 onto disc 3
15	1111	Move disc 1 onto disc 2 (final move)

III. THE SET OF NATURAL NUMBERS

The set of natural (or counting) numbers is defined in this outline as: $\{1,2,3 \ldots\}$

Properties of the Natural Numbers

Concept: The natural numbers are closed under addition and multiplication.

In working with the natural numbers we find certain things to be always true. First, if we add two natural numbers, the sum is always a unique natural number. This property is called closure; we say the set of natural numbers is closed with respect to addition. There are sets of numbers which are not closed under addition; for example, the set of odd numbers $\{1,3,5,7 \ldots\}$ is not closed under addition because $1+7$ is not a member of the set of odd numbers. The set of natural numbers is *not* closed with respect to subtraction, for $8-9$ is not a natural number. The set of natural numbers is also closed with respect to multiplication; for example, $4 \times 6 = 24$, a natural number. The set of multiples of 5, $\{5,10,15 \ldots\}$ is closed with respect to multiplication but the set of natural numbers <10 is not closed under multiplication.

Concept: Addition and multiplication of natural numbers is commutative.

The next property to be discovered is that if we add two natural numbers, it does not matter in what order we do it; to illustrate, $3+4=4+3$. This is the commutative property and we say addition is commutative (reversible).

Multiplication is also commutative, $3 \cdot 5 = 5 \cdot 3$. There are also operations which are not commutative; for example, division and subtraction are not, for $\frac{6}{2} \neq \frac{2}{6}$ and $4-1 \neq 1-4$.

Concept: Addition and multiplication of natural numbers is associative.

The next property we are aware of is the associative property. To add the three numbers $4+2+3$, we are using a binary operation and, therefore, add them two at a time. We then have the choice of adding

$(4+2) + 3$	or	$4 + (2+3)$
$6 + 3$		$4 + 5$
9		9

In either case the sum is the same and we say that addition is associative or the associative property holds for addition. In general for any natural numbers a, b, and c, $(a+b)+c = a+(b+c)$. As we might imagine multiplication is associative.

$$\begin{aligned}(2\cdot6)\cdot3 &\overset{?}{=} 2\cdot(6\cdot3)\\ 12\cdot3 &\overset{?}{=} 2\cdot18\\ 36 &= 36\end{aligned}$$

Notice that when we use the associative property, the order of the numbers stays the same. Thus

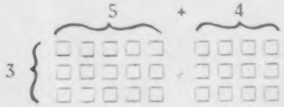
To Prove: $(3+4)+6 = 3+(6+4)$

$$\begin{aligned}(3+4)+6 &= (3+4)+6 &&\\ (3+4)+6 &= 3+(4+6) && \text{associative property}\\ (3+4)+6 &= 3+(6+4) && \text{commutative property}\end{aligned}$$

Neither subtraction no division is associative.

Concept: Multiplication is distributive over addition for the natural numbers.

The distributive property involves both multiplication and addition; multiplication is distributed over addition. Thus, $3\cdot(5+4) = (3\cdot5) + (3\cdot4)$ or $(3\cdot5) + (3\cdot4) = 3\cdot(5+4)$ represent the distributive law. A picture of this is shown below.



The general form of the distributive law is $a\cdot(b+c) = a\cdot b + a\cdot c$ or $a\cdot b + a\cdot c = a(b+c)$

This postulate is probably the most frequently used (consciously, at least) of all the postulates. Below are four applications of the distributive property. (The algebraic applications in the illustrations (1) and (2) are intended for 8th grade use.)

(1) In multiplying by a one-digit numeral:

$$\begin{array}{r}12\\ \underline{\times 4}\\ 48\end{array}\quad \text{really means } 4\cdot(12) = 4\cdot(10+2) = \\ = (4\cdot10) + (4\cdot2)\\ = 40 + 8 = 48$$

Algebraically this is analogous to multiplying a binomial by a monomial.

$$x\cdot(y+3) = x\cdot y + x\cdot3 = xy + 3x$$

(2) In multiplying two two-digit numerals:

$$\begin{array}{rl} 38 & \\ \underline{\times 24} & \\ 32 & = 4\cdot 8 \\ 120 & = 4\cdot 30 \\ 160 & = 20\cdot 8 \\ \underline{600} & = 20\cdot 30 \\ 912 & \end{array} \qquad \text{or} \qquad \begin{aligned} 24\cdot 38 &= (20+4)\cdot(30+8) \\ &= (20+4)\cdot 30 + (20+4)\cdot 8 \\ &= 30\cdot(20+4) + 8(20+4) \\ &= (30\cdot 20)+(30\cdot 4)+(8\cdot 20)+(8\cdot 4) \\ &= \quad 600 \;+\; 120 + 160 \;+\; 32 \\ &= 912 \end{aligned}$$

Algebraically this would be analogous to multiplying two binomials such as:

$$\begin{aligned} (ax+b)\ (cx+d) &= (ax+b)\cdot cx + ax+b)\cdot d \\ & \quad cx(ax+b) + d(ax+b) \\ & \quad acx^2 + bcx + adx + bd \\ & \quad acx^2 + (bc+ad)x + bd \end{aligned}$$

(3) In factoring we use the reverse form: $a\cdot b + a\cdot c = a\cdot(b+c)$ Notice that a is a factor of $a\cdot b$ and also a factor of $a\cdot c$.

(4) In division, we think of $8 \div 2$ or better $\frac{8}{2} = \boxed{\ ?\ }$ as $2\cdot\boxed{\ ?\ } = 8$.

In another example $\overset{?}{2\overline{)\,320}}$, this is the same as $\frac{320}{2} = \boxed{\ ?\ }$

$\frac{300+20}{2} = \boxed{\ ?\ }$ In other words $2\cdot\boxed{\ ?\ } = 300+20$

The answer to $\frac{320}{2}$ is 150+10 or 160.

In $\frac{484}{4} = \frac{400+80+4}{4}$ we can divide each term by 4; in other words, find a common factor of 4 in each term, if we can, then

$$\begin{aligned} \frac{4(100+20+1)}{4} &= \frac{4}{4}\cdot(100+20+1) \\ &= 100 + 20 + 1 = 121 \end{aligned}$$

Concept: The identity element for multiplication of natural numbers.

Another important property of the natural numbers is that there is only one natural number which when multiplied by any second natural number results in a product which is identical to the second number. Of course this number is 1, for if n is any natural number, then $n\cdot 1 = n$. The number 1 is therefore called the identity element for multiplication *(in the natural numbers there is no identity element for addition because there is no zero in the natural* *numbers).* From the identity above we can also write

$$\frac{n}{n} = n \div n = n\cdot\frac{1}{n} = \frac{1}{n}\cdot n = 1;$$ where $\frac{1}{n}$ is the reciprocal of n. Notice any number multiplied by its reciprocal is one. Of course there is no number $\frac{1}{n}$ in the set of natural numbers so this relation finds its use later. However, there is no number $\frac{1}{n}$ when $n = 0$. It is also true that $n^1 = n$ and $1^a = 1$.

Concept: The successor of a number.

An important idea is the concept that each natural number has a successor which is obtained by adding 1; in other words, $n+1$ is the successor of n or the next higher consecutive natural number. The identity element is very important in division of the rational numbers.

Concept: The cancellation law for natural numbers.

In order that we do not produce negative numbers and fractions when we operate on the natural numbers, we postulate the cancellation laws. For addition the cancellation law is:

if $a+b = c+b$, then $a = c$.

For multiplication the cancellation law is:

if $a\cdot b = c\cdot b$, then $a = c$.

(Notice that when we get to the integers, which include the negative whole numbers, we will not have to postulate the cancellation law for addition).

Concept: The natural numbers are ordered.

The final postulate for the natural numbers is that of order: for any two elements a and b, one and only one of the three following statements holds

(1) $a = b$
(2) There is a natural number x such that $a+x = b$
(3) There is a natural number y such that $a = b+y$

This is another way of saying either:

(1) $a = b$, *or* (2) $a<b$, *or* (3) $a>b$. In other words there is an order in which the natural numbers can be arranged.

Let us summarize the properties of the natural numbers.

Under Addition	Under Multiplication
Closure property: $a+b$ is a unique natural number	Closure property: $a \cdot b$ is a unique natural number
Commutative property $a+b=b+a$	Commutative property $a \cdot b = b \cdot a$
Associative property $a+(b+c) = (a+b)+c$	Associative property $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Cancellation law: if $a+b = c+b$ then $a = c$	Cancellation law: if $a \cdot b = c \cdot b$ than $a = c$
	Identity element: there is a number 1 such that $a \cdot 1 = 1 \cdot a = a$

Distributive property: $a(b+c) = a \cdot b+a \cdot c$ $(b+c) \cdot a = ba+ca$
Order: $a = b$ or $a<b$ or $a>b$

IV. THE SET OF WHOLE NUMBERS

The set of whole numbers is defined as the set whose elements are the natural numbers and zero, {0,1,2,3, ...}

Concept: The four operations on whole numbers are binary.

An operation in mathematics is usually undefined except to say it is a use of elements in our set to designate *another element in the set. A binary operation* is an operation on just two elements. Addition is a binary operation; for example, in 3 + 7 + 5 we add 7 to 3, then add 5 to the sum, even though we might do this mentally; multiplication is also a binary operation. Taking a square root is not a binary operation but a unary operation because we are operating on only one element.

Since a mathematical system says nothing about the set of elements being numbers, the operations do not have to be addition and multiplication as we know them, and these operations do not necessarily have to relate to each other (as the operations of addition and multiplication on the natural numbers do).

We can think of addition as a combining of two sets of objects as in $\underbrace{oo}_{2} + \underbrace{oooo}_{4} = \underbrace{oooooo}_{6}$ or we can also think of addition as counting

$$\underbrace{11}_{2} + \underbrace{1111}_{4} = \underbrace{\overset{1\ 2\ 3\ 4\ 5\ 6}{111111}}_{6}$$

and our common multiplication is usually thought of as a repeated addition of like addends, as in 2x8 = 8+8 = 16. But again, multiplication can be defined by the multiplication table and not thought of as being related to addition at all. Multiplication under the set of rational numbers cannot be thought of as repeated addition ($\frac{1}{4} \times \frac{2}{3} = ?$)

Concept: Some operations have inverses.

Sometimes we do things, then undo them. For example, we open a door, then close it. We turn on a switch, then turn it off. The second operation is the inverse of the first one. The inverse of opening a book is closing it. *What is the inverse of addition?* Subtraction. The inverse of putting on your shoes is taking off your shoes. *The inverse of "putting on your shoes" is not "not putting on your shoes"* (this would be the negation).

Select the phrases below that describe operations which have inverses and write the inverses.

(a) Putting on your coat
(b) Eat an apple (no inverse)
(c) Take a step forward
(d) Smell a flower
(e) Multiply
(f) Take off your hat
(g) Add
(h) Jump from a plane in flight

Answers:
(a) Taking off your coat
(c) Take a step backward
(e) Divide
(f) Put on your hat
(g) Subtract

Concept: Subtraction is not considered a basic operation but is defined as the inverse of addition.

Thus $8-3 = \boxed{}$ means $3 + \boxed{} = 8$. Note that $3 - 8$ cannot be done for the natural or whole numbers.

The relation between subtraction and its inverse, addition, is used to check subtraction. To check $10 - 2 = 8$ we write an addition.

Thus, in $8 + 2 = 10$ the result is the number we started with. This is how we check our subtraction, by addition. This horizontal form should be used more often in all our work. We also think of subtraction as a process of taking away.

Concept: Division also is not considered a basic operation, but is defined as the inverse of multiplication.

Thus $\frac{8}{2} = \boxed{}$ means $2 \times \boxed{} = 8$. Note that $\frac{8}{3} = \boxed{}$ cannot be done for the natural or whole numbers.

If $8 \times 3 = 24$
then $\frac{24}{3} = 8$

How else do we think about division? As a repeated subtraction; for example,

$$\begin{array}{r} ? \\ 3\overline{)\ 12} \end{array} \qquad \begin{array}{r} 12 \\ \underline{-\ 3} \\ 9 \\ \underline{-\ 3} \\ 6 \\ \underline{-\ 3} \\ 3 \\ \underline{-\ 3} \\ 0 \end{array}$$

4 groups of 3 in 12

∴ 12 divided by 3 is 4.

Write each of the following using the inverse operation.

(a) $18 \div 6 = \square$ $\boxed{6 \times \square = 18}$

(b) $12 \cdot \square = 36$ $\boxed{\frac{36}{12} = \square}$

(c) $\frac{56}{\square} = 7$ $\boxed{\square \cdot 7 = 56}$

(d) $\frac{\square}{14} = 42$ $\boxed{14 \cdot 42 = \square}$

How do we read a division example? $3\overline{)\,12}$ is read:

> *12 divided by 3 equals what?*
> *What times 3 equals 12?*
> *3 is contained in 12 how many times?*

We can write a division four ways as

(1) $8 \div 2$ (2) $\frac{8}{2}$ (3) $2\overline{)\,8}$ (4) 8:2 (ratio). Of these ways, the fractional form $\frac{8}{2}$ is most important because we use this form in writing number sentences later on. When the division $\frac{8}{2}$ is read, it should be read as "8 divided by 2" or "*what times 2 equals 8?*" Reading it as how many 2's are contained in 8 is correct but reading it as "2 goes into 8" leads to complications.

Probably one of the areas in arithmetic which pupils (other than the mathematically gifted) find most troublesome is long division. One of the reasons is the 4-step algorithm, which the pupils do not understand (1 divide, 2 multiply, 3 subtract, 4 bring down).

$$\begin{array}{r} 2 \\ 30 \\ \underline{400} \\ 2\overline{)\,864} \\ \underline{-800} \\ 64 \\ \underline{-60} \\ 4 \\ \underline{-4} \\ 0 \end{array}$$

Look at the example on the left. We can think $\square \cdot 2 = 864$ or how many 2's are contained in 864. In either case we can say about 400 2's or $864 \div 2 =$ about 400. Now $400 \cdot 2 = 800$, leaving 64 still to be divided by 2. $64 \div 2 =$ about 30 and $30 \cdot 2 = 60$ leaving 4 to be divided by 2. This last division is $\frac{4}{2} = 2$; the partial quotients are added to give the final quotient of $400+30+2 = 432$. This is much more meaningful than the older 4-step process.

$$\begin{array}{r} 7 \\ \underline{20} \\ 32\overline{)864} \\ \underline{-640} \\ 224 \\ \underline{-224} \\ 0 \end{array}$$

Had we divided 864 by 32, we could estimate the partial quotient or $\square \cdot 32 = 864$, say about 20 because $20 \cdot 30 = 600$. Then $20 \cdot 32 = 640$ leaving 224 to be divided by 32. $224 \div 32 = 7$ and $7 \cdot 32 = 224$. The final quotient is $20+7 = 27$.

Divide in each example below using partial quotients (in each case estimate answer first).

(a) $18\overline{)432}$ (b) $37\overline{)4440}$ (c) $17\overline{)2982}$

Answers:

$$\text{(a)}\quad \begin{array}{rr} & 4 \\ & 20 \\ \cline{2-2} 18) & 432 \\ & -360 \\ \cline{2-2} & 72 \\ & -72 \\ \cline{2-2} & 0 \end{array} \qquad \text{(b)}\quad \begin{array}{rr} & 20 \\ & 100 \\ \cline{2-2} 37) & 4440 \\ & -3700 \\ \cline{2-2} & 740 \\ & -740 \\ \cline{2-2} & 0 \end{array} \qquad \text{(c)}\quad \begin{array}{rr} & 5 \\ & 70 \\ & 100 \\ \cline{2-2} 17) & 2982 \\ & -1700 \\ \cline{2-2} & 1282 \\ & -1190 \\ \cline{2-2} & 92 \\ & -85 \\ \cline{2-2} & 7 \end{array}$$

432:18 = 24

4440:37 = 120

2982:17 = 175 R 7
$= 175\frac{7}{17}$

Concept: The postulates listed for the natural numbers hold for the whole numbers. However, the inclusion of 0 gives us an identity element for addition.

Zero added to any whole number results in that same whole number.

i.e., $a+0 = 0+a = a$.

Zero can usually be handled easily in the operations of addition and multiplication and in the inverse operation of subtraction to give:

(1) $a+0 = a$
(2) $a-0 = a$
(3) $a-a = 0$
(4) $a\times0 = 0 = 0\times a$
(5) $0\times0 = 0$

Let us look at the division involving zero in terms of the inverse operation multiplication.

Thus $\frac{0}{6} = \boxed{?}$ means $6\times\boxed{?} = 0$. From (4) above the only solution is zero; therefore $\frac{0}{6} = 0$, or (6) $\frac{0}{a} = 0$

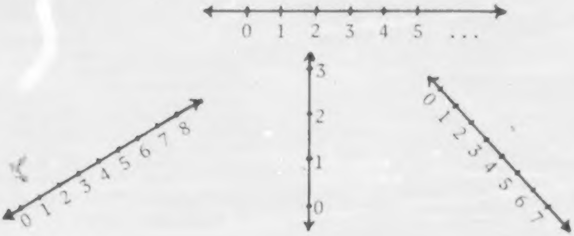
Concept: The denominator of a fraction may not be zero.

Now for the example $\frac{7}{0} = \boxed{?}$; this means $0\cdot\boxed{} = 7$. Since by (4) above $0\times a$ must equal zero, there is no number which will make this sentence true. We say then (7) $\frac{a}{0}$ = no answer or impossible or we cannot divide by zero. Whenever we see a general divisor in the literature, we see $n\neq0$, which says that we cannot divide by zero.

The other possibility in a division would be $\frac{0}{0} = \boxed{}$; this means $0 \times \boxed{} = 0$. From (4) and (5) above, 0,1,2,..., any number will make this sentence true; there is no unique answer. We say then (8) $\frac{0}{0}$ = any number or is indeterminate or undefined.

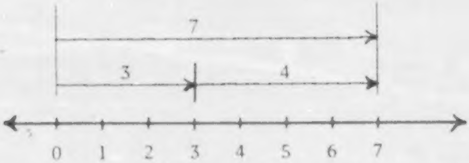
The Number Line

The number line is a geometric representation of each number by a point on a line. There must be the same distance between the points representing successive whole numbers since the difference between the successive whole numbers is the same (one). The arrowheads at the ends of the number line are used to show the line can continue to the right and left (consideration of the numbers to the left of zero may be deferred until the 8th grade unit on integers.) The number line is usually horizontal, but may be vertical or oblique as well.

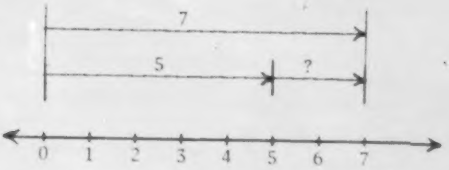


Concept: Picturing addition on the number line.

Show 3+4 on the number line.

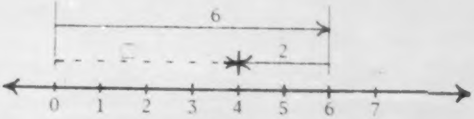


If we start at zero, go to the right 3, then to the right 4, we have gone to the right 7. To avoid the static nature of the numbers here, the 3 and 4 (positive integers) may be thought of as instructions to move to the right. The negative integers would mean to move to the left. *Show 5+ $\boxed{}$ = 7 on the number line.*

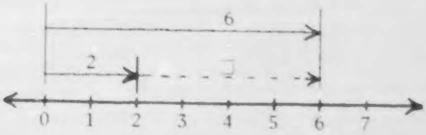

7
5
?
0
1
2
3
4
5
6
7

Concept: Picturing subtraction on the number line.

Show 6 - 2 = □ on the number line:


6
□
2
0
1
2
3
4
5
6
7

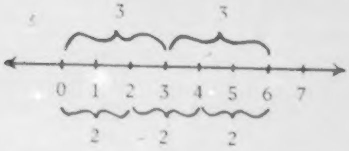
or 2 + □ = 6 as the inverse would look like this:


6
2
□
0
1
2
3
4
5
6
7

Concept: Picturing multiplication on the number line.

Show 2x3 and 3x2 on a number line:

2x3 = 6

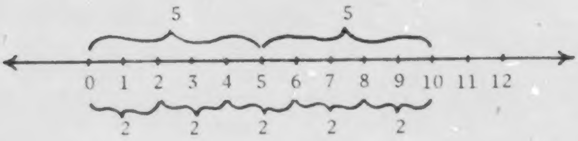

3
3
0
1
2
3
4
5
6
7
2
2
2

3x2 = 6

Concept: Picturing division on the number line.

Show 10÷5 and 10÷2 on a number line.

10÷5 = 2



10÷2 = 5

The braces above the number line show by successive subtractions that 2 groups of 5 are included in 10, the braces below the number line show by successive subtractions that 5 groups of 2 are included in 10.

Since there are various ways in rounding numbers, the method of rounding whole numbers is left to the discretion of the teacher.

Mathematical Sentences

A mathematical sentence is a statement using mathematical symbols, and may be either true or false.

i.e. 8+5 = 13 a true statement
7+9 = 17 a false statement

At this level, the definition of an *open sentence* should be simply, a mathematical sentence which contains a *variable* or *placeholder*. An open sentence is either true or false depending upon the numbers chosen to take the place of the variable.

i.e. $6 + \square = 11$ *true* if $\square = 5$, because 6+5 = 11
false if $\square = 8$, because 6+8 = 14

The replacement set is the set from which these numbers may be chosen. If there are numbers in the replacement set which make the sentence true, this set of numbers is called the *solution set;* if there are no replacements that make the sentence true, then the null set is called the solution set. The solution set is a subset of the replacement set. It makes a difference from what set we can draw our solution set. Suppose we have the sentence $7 + \square = 5$ and we want to find the solution set. If we are talking about the natural numbers only, the solution set is $\emptyset$, the empty set or null set, because there is no natural number which when put in place of the box (or in the box) will make this sentence true. However, if we can draw from the set of integers our solution set would be $\{-2\}$. The set from which we can draw our solution set is called the replacement set. Whenever we work with open sentences we must then specify the replacement set.

An identity is an open sentence which is true for all permissible replacements of the variable(s). Examples of identities are:

(a) $\square + 3 = 3 + \square$ (If we substitute a 4 in the box on the left hand side, we must also substitute a 4 in every box in the sentence.)

(b) $\square + \Delta = \Delta + \square$

(c) $3 \cdot (y+5) = 3 \times y + 3 \times 5$

(d) $\square + 0 = 0$

(e) $\Delta \times 1 = \Delta$

Note: One use of frames such as $\square$, Δ, and ∇ as place holders has been recommended by leading nation-wide studies for elementary school pupils and for the slower pupils at the junior high level. With classes of higher ability in the junior high school, letters may also be introduced as placeholders or variables as in (c) above.

Tell which of the following sentences are true, false or open.

(a) $6+3 = 9$

(b) $\square + 8 = 10$

(c) _________ sits in the first row.

(d) $7+5 = 11$

(e) $n+8 = 20$

(f) $3 \times 2 = 7$

(g) ________ is the tallest student in the class.

(h) $6+2 = \frac{24}{3}$

Answers:

(a) true

(b) open

(c) open

(d) false

(e) open

(f) false

(g) open

(h) true

Students should become familiar with the following symbols and their meanings when read from left to right:

$=$ means "is equal to"

$\neq$ means "is not equal to"

$>$ means "is greater than"

$\geq$ means "is greater than or equal to"

$<$ means "is less than"

$\leq$ means "is less than or equal to"

Practice should be given in reading mathematical sentences containing these symbols; such as, $5+2 = 7$, $2+4 \neq 3+5$, $4+3>5$, $7-4<6+2$, $6 \leq a$, $10 \geq b$. A continued inequality, such as, $2+1<5<4+3$ is most easily read from the middle as: five is greater than (2+1) *and* less than (4+3).

Divisibility

A whole number is divisible by a second whole number, if the first can be divided by the second whole number without a remainder. The rules (for base ten) which should be discussed at this time are:

1) A number is divisible by two if the last digit of the numeral is an even number.
2) A number is divisible by three of the sum if the digits is divisible by three.
3) A number is divisible by five if the last digit is a five or a zero.
4) A number is divisible by ten if the last digit is zero.

These rules are helpful in complete factorization.

As enrichment the following divisibility rules may be introduced:

. A number is divisible by 4 if the number formed by the digits in the units and tens places is divisible by 4.

. A number is divisible by 6 if the number is divisible by both 2 and 3.

. A number is divisible by 7 if the difference between twice its unit's digit and the number formed by its non-unit digits is divisible by 7. *Is 693 divisible by 7?* For 693, 3 is the unit's digit and 69 would be the number formed by the non-unit digits of 693. Then, twice 3 is 6; $69-6 = 63$ which is divisible by 7. Therefore 693 is divisible by 7.

If a number is divisible by 4 and 2, is it necessarily divisible by 8? 28 is divisible by 4 and 2 but not divisible by 8. The rule for 8 is similar to that of 4 except that 1000 or any number of thousands is divisible by 8.

. A number is divisible by 8 if the number formed by the units, tens, and hundreds digits is divisible by 8.

The proof for the divisibility by 9 is rather simple. Suppose we take the number 432. This may be written

$$\begin{gathered} 4 \cdot 100 + 3 \cdot 10 + 2 \cdot 1 = \\ 4(99+1) + 3(9+1) + 2 = \\ (4 \cdot 99) + 4 + (3 \cdot 9) + 3 + 2 = \\ (4 \cdot 99) + (3 \cdot 9) + (4+3+2) \end{gathered}$$

Since (4·99) and (3·9) are divisible by 9 and (4+3+2), the sum of the digits, is divisible by 9, the original number is divisible by 9. Notice also that if the sum of the digits of a number were not 9, or a multiple of 9, this sum would be the remainder when the number is divided by 9. In general, then

$$\begin{aligned} &100h + 10t + u = \\ &(99+1)h + (9+1)t + u = \\ &99h + h + 9t + t + u = \\ &(99h+9t) + h + t + u, \end{aligned}$$

the part in parentheses is divisible by 9, leaving the sum of the digits of the original number. The rule then for 9:

- A number is divisible by 9 if the sum of its digits is divisible by 9.

The divisibility rule for 9 is the basis for the method of checking by casting out 9s.

- A number is divisible by 11 if the difference between the sum of the digits in the odd places (units, hundreds, and so forth) and the sum of the digits in the even places (tens, thousands, and so forth) is zero. Thus 4796 is divisible by 11 because

7+6 = 13

sum of digits in the odd places

13-13 = 0

4+9 = 13

sum of digits in the even places

- A number is divisible by 13 if the sum of four time the unit's digit added to the number formed by dropping the unit's digit is divisible by 13. The process may be repeated several times.

Thus 490 is not divisible by 13 because 49 + (4x0) = 49 is not divisible by 13. 195 is divisible by 13 because 19 + (4x5) = 39 which is divisible by 13.

- It is possible to test the divisibility by 7, 11, and 13, each, any, or all, by considering the number to base 1000 (the usual grouping by sets of 3 digits) and using a test like that for divisibility by 11. See The Arithmetic Teacher, April 1961.

Primes

A *prime number* is a whole number greater than one, which is divisible only by itself and one. A *composite number* is a whole number greater than one, which is not prime.

One interesting way of arriving at the prime numbers <100 is by using the Sieve of Eratosthenes. *Can you guess how the Sieve of Eratosthenes works?* Cross out 1 then all the composite numbers. This will leave the primes. The crossing out should be done systemmatically. Cross out 1, not 2, but all the multiples of 2, not 3, but all the multiples of 3, not 4, not 5, but the multiples of just the prime numbers. The prime numbers <100 as seen below are:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37
41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

~~1~~	②	③	~~4~~	⑤	~~6~~	⑦	~~8~~	~~9~~	~~10~~
⑪	~~12~~	⑬	~~14~~	~~15~~	~~16~~	⑰	~~18~~	⑲	~~20~~
~~21~~	~~22~~	㉓	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	㉙	~~30~~
㉛	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	㊲	~~38~~	~~39~~	~~40~~
㊶	~~42~~	㊸	~~44~~	~~45~~	~~46~~	㊼	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	(53)	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	(59)	~~60~~
(61)	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	(67)	~~68~~	~~69~~	~~70~~
(71)	~~72~~	(73)	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	(79)	~~80~~
~~81~~	~~82~~	(83)	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	(89)	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	(97)	~~98~~	~~99~~	~~100~~

The *prime factors* of a number are all the factors of the number that are also prime numbers. A number expressed as the product of primes is the *complete factorization* of that number. Although there are only two prime factors of 56, 2 and 7, the prime factorization of 56 must be stated 2 x 2 x 2 x 7.

Practice should be given to point up the difference between factors of; prime factors of; and complete factorization of a number. Thus, (1) factors of 12 = {2,6}, {3,4}, {1,12}; (2) prime factors of 12 = {2,3}; (3) complete factorization of 12 = 2 x 2 x 3, which may be stated as express 12 as a product of primes.

Concept: The greatest common factor of a set of numbers is the largest factor that is common to the set.

It can be identified by writing the factors of each number and circling the largest common factor.

Find the greatest common factor of 12, 16, 28.

The factors of 12 = 3 x ④ = 6 x 2 = 12 x 1;

The factors of 16 = 4 x ④ = 8 x 2 = 16 x 1
The factors of 28 = 14 x 2 = 7 x ④ = 28 x 1

The greatest common factor is 4.

Two numbers are said to be *relatively prime* to each other if heir greatest common factor is one.

hen, 15 and 8 are relatively prime since:

15 = 5 x 3 = 15 x ①
8 = 4 x 2 = 8 x ①

and the greatest common factor is 1

If any number is multiplied by a whole number other than zero, he product is said to be a *multiple* of the given number.

.e. 3 x 6 = 18, 18 is a multiple of 3 (and 6)

oncept: The least common multiple of a set of numbers is the smallest common multiple of the set.

(1) Consider the set {3,5}. List the multiples in a horizontal ine:

Multiples of 3 ⟶	3	6	9	12	⑮
Multiples of 5 ⟶	5	10	⑮	20	25

en, 15 is the least common multiple of {3,5}

(2) Consider the set {3,4,8}

ltiples of 3 ⟶	3	6	9	12	15	18	21	㉔
ltiples of 4 ⟶	4	8	12	16	20	㉔	28	
ltiples of 8 ⟶	8	16	㉔	32	40	48	56	

en, 24 is the least common multiple of {3,4,8}

A student could make a card with the first twelve multiples of o 12 in a horizontal arrangement to use for finding least common ltiples, as well as for finding least common denominators. This often a stumbling block for him in addition and subtraction of ionals.

Order of Operations

Whenever there is more than one operation indicated we must ee upon the order in which they are to be done. This agreement order of operations is the following: Multiplication and ision are done first, then addition and subtraction (from left ight). If we use parentheses or some other symbols of inclusion, we consider what is within these symbols as one number and perform the operation within the parentheses first. If we use more than one symbol of inclusion we start from the inner one first. In giving examples to the students, be careful not to get too involved with negative numbers.

(a) $8x5-12:4+5=\boxed{}$

$40-3+5=$

$37+5=$ Answer: 42

(Notice here we work from left to right on the second line; we do not say 3+5 first then 40-8)

(b) $12:4:2=\boxed{}$

$3:2=$ Answer: $\frac{3}{2}$ or $1\frac{1}{2}$

(c) $[4+\{(8x6)-3\}]=\boxed{}$

$[4+\{48-3\}]=$

$[4+45]=$ Answer: 49

(d) $4x10^3+2x10^2+5x10^1+6x10^0=\boxed{}$

$4000+200+50+6=$ Answer: 4256

(e) $\frac{15x3-9}{10+2}+14:7=\boxed{}$

$\frac{45-9}{12}+2=$

$\frac{36}{12}+2=$

$3+2=$ Answer: 5

(f) $4+3x6-18:2=\boxed{}$ Answer: 13

(g) $12:6+2x5-7=\boxed{}$ Answer: 5

(h) $2(3+9)-6x1+3=\boxed{}$ Answer: 21

(i) $[\{6+(3\cdot2)\}-(36:9)]\ x\ 2=\boxed{}$ Answer: 16

The following material may be used to stimulate further interest in mathematics. It may best be used for enrichment.

Magic Squares

The importance of the magic squares at this level does not lie in making them, but in using them as motivation for working with the fundamentals.

A magic square consists of different numbers arranged in the form of a square so that the sum of the numbers in every row, column, and diagonal is the same. Figure 1 shows a magic square (the simplest) which contains 9 *cells* (small squares) and is said to be of *order* 3 (3 rows, 3 columns). The integers happen to be consecutive, beginning with one.

8	1	6
3	5	7
4	9	2

Figure 1

The evidence of the first magic square goes back to China about 2200 BC and looked like this:

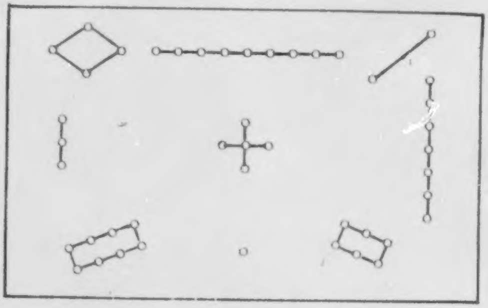
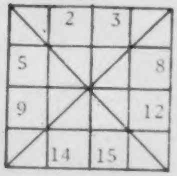


Figure 2

which of course is the same one as in figure 1 except it has been inverted. The magic square below in figure 3 appeared in a very famous engraving called Melancholia by Durer in 1514. This magic square has some remarkable qualities, such as: the date the painting was done, 1514, is at the bottom of the square.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Figure 3



	2	3	
5			8
9			12
	14	15	

Figure 4

The magic square of order 4 or 4n is made this way. Draw the two diagonals of the square. In placing the numbers in the cells the first time, do not place one in any cell that lies along the diagonals. Start by counting from 1 in the upper left hand corner and completing each row from left to right. One is not placed because this cell lies on a diagonal, the 2 goes in, the 3 goes in, but not the 4. The 5 goes but not the 6 or 7 (those cells lie on the diagonal). The 8 goes in, the 9, but not 10 or 11. The 12, 14, and 15, but not the 13 or 16 (see figure 4). Now starting in the upper left hand corner again and working from 16 down, place the remaining numbers in the cells the same way. The resulting magic square whose sum is 34 is shown in figure 5. If we start with 1 and use consecutive integers, the sum of every row, column and diagonal for a magic square of order n is $\frac{n^3 + n}{2}$. For the magic square in figure 5, $\frac{4^3 + 4}{2} = 34$.

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

Figure 5

Questions such as these may be asked of the students.

(a) *Fill in the cells below to make the figure a magic sqaure.* (The numbers are arranged so that the pupils will not recognize any order, but will have to use the fact that the sum of one diagonal, therefore the sum of all rows, columns and diagonals, will be known. They can then find the missing numbers in a row given four of the five numbers in that row.)

30	32	44		18
16		40	42	4
	14	26	38	
48	10			36
34		8	20	

Answer:

30	32	44	6	18
16	28	40	42	4
2	14	26	38	50
48	10	12	24	36
34	46	8	20	22

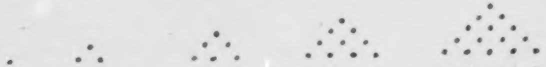
The magic square above was made by starting with 2, using just the consecutive even numbers, rotating the magic square 90 degrees counterclockwise, then removing enough numbers to still make it possible to complete.

(b) *Is the following a magic square?* No. This is not a magic square because the sum of every row, column and diagonal is not 34. (The 10 and 1 were interchanged in a transposed magic square derived from figure 5).

9	6	3	16
4	15	1	5
14	10	8	11
7	12	13	2

Figurate Numbers

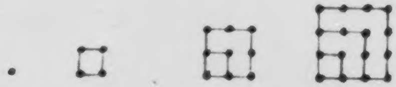
Another interesting topic related to addition of whole numbers is that of figurate or polygonal numbers. Since the triangle is the simplest polygon, we will look at the triangular numbers.



1 =	1+2 =	1+2+3 =	1+2+3+4 =	1+2+3+4+5 =	. . .
1	3	6	10	15	. . .

The triangle numbers are a sequence of numbers each equal to a finite series of natural numbers starting with one. (Note: A sequence is an ordered set of quantities; for example, the triangular numbers and the whole numbers. A series is an indicated sum of an ordered finite or infinite set of terms; for example, 1+3+5 and 1+2+3+... are series.

The square numbers are equal to the series of odd numbers. Thus the sequence {1,4,9,16,25...} which makes up the set of square numbers has the following equations: 1 = 1, 4 = 1+3, 9 = 1+3+5, 16 = 1+3+5+7, ...



1 = 1+3 = 1+3+5 = 1+3+5+7 = . . .

1 = 4 = 9 = 16 =

1^2 2^2 3^2 4^2

The sum of two consecutive triangular numbers (6+10) is a square number (16) as shown in figure 9. The pentagonal numbers are formed by the series of every third integer, starting with one.

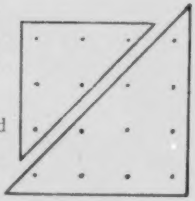
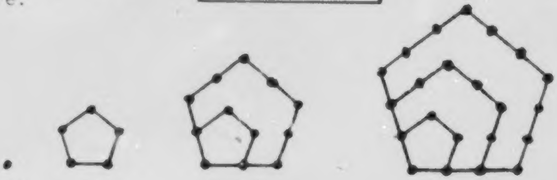


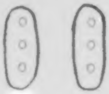
Figure 9



1 = 1+4 = 1+4+7 = 1+4+7+10 = . . .

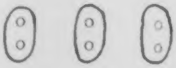
1 5 12 22

What do we mean by multiplication? What does 2x3 mean? Multiplication is a short way of adding. 2x3 means 2 threes or 3 + 3 = 6. The students should see the picture:



3 + 3 = 6

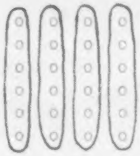
Notice the picture above is not the same as the one for 3x2 or 3 twos, which would look like this:



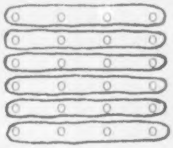
2 + 2 + 2 = 6

The product is the same but the picture is different.

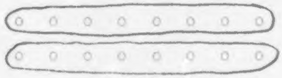
Draw a picture of 4x6.



Draw a picture of 6x4.

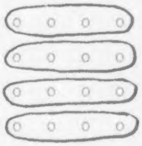


Draw a picture of 2x8.

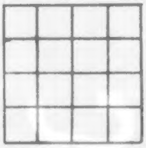


2 times a group of 8 circles.

Draw a picture of 4x4 = 16.



We could also draw these
as little squares instead of circles:



Missing Digits

Restoration problems are those which involve missing digits in an arithmetical example.

Restoration problems may be given in the form below. In this type, two, or more than two missing digits in one column give more than one solution:

```
(a)    6?49      6749       (b)  ?62       562
      +?24?     +9240           394?      3943
      ?59?9     15989           ?8?7      8807
                                ?3312     13312
```

Alphametics

Alphametics are puzzles given as an exercise with digits replaced by letters; each letter represents a single digit and the same letters represent the same digits. Find the numbers such that:

```
  H Ø C U S    We will assume whole numbers only are involved
+ P Ø C U S    although this addition has three decimal solutions.
P R E S T Ø    In solving this type of puzzle it is advisable to
```

lay out a plan of procedure, rather than try hit or miss methods. Let's solve this puzzle.

$$\begin{array}{r} \_\;\_\;\_\;\_\;\_ \\ +\ 1\;\_\;\_\;\_\;\_ \\ \hline 1\;\_\;\_\;\_\;\_\;\_ \end{array}$$

Upon observation, we see P<2. If P = 0, we will obtain the decimal solutions, which are possible, but with which we are not concerned at this point. We will assume that P = 1 then. If P = 1, H = 9 or H = 8. If H = 9 there can be no ten thousands from the processing column because it would make R = 1 which cannot be because P = 1. If H = 9, R = 0. Since S gives us 3 digits in our solution we will now try to find what S represents. We must actually place each digit in its corresponding blank as we work along.

$$\begin{array}{rcccccc} & & \text{H} & \text{Ø} & \text{C} & \text{U} & \text{S} \\ + & & \text{P} & \text{Ø} & \text{C} & \text{U} & \text{S} \\ \hline & \text{P} & \text{R} & \text{E} & \text{S} & \text{T} & \text{Ø} \end{array}$$

$$\begin{array}{r} 9\;\_\;\_\;\_\;\_ \\ +\ 1\;\_\;\_\;\_\;\_ \\ \hline 1\;0\;\_\;\_\;\_\;\_ \end{array}$$

S ≠ 0 for R = 0
S ≠ 1 for P = 1

If S = 2 then Ø = 4, E = 8. (E ≠ 9 since H = 9). But then C+C could equal neither 12 nor 2, ∴ S ≠ 2.

If S = 3 then Ø = 6, with one ten thousand for the following column which we showed cannot be, ∴ S ≠ 3

If S = 4 then Ø = 8, with one ten thousand for the following column which we showed can be, ∴ S ≠ 4

If S = 5, then Ø = 0. But R = 0. ∴ S ≠ 5

S = 6, Ø = 2 then Ø + Ø with one thousand from the previou column, gives E = 5, C = 8, U = 3 T = 7 and a solution. (If E = 4, then C = 3 and no value of U will work).

$$\begin{array}{cccccc} & \text{H} & \text{Ø} & \text{C} & \text{U} & \text{S} \\ & \text{P} & \text{Ø} & \text{C} & \text{U} & \text{S} \\ \hline \text{P} & \text{R} & \text{E} & \text{S} & \text{T} & \text{Ø} \end{array} \quad \Bigg| \quad \begin{array}{cccccc} & & 1 & & 1 & \\ & 9 & 2 & 8 & 3 & 6 \\ & 1 & 2 & 8 & 3 & 6 \\ \hline 1 & 0 & 5 & 6 & 7 & 2 \end{array}$$

If S = 7 then Ø = 4, E = 8, C = 3 with one hundred from the previous column, then no value of U will work. ∴ S ≠ 7

If S = 8, then Ø = 6 gives one ten thousand more which we cannot have, ∴ S ≠ 8 S ≠ 9 for H = 9.

The same type of analysis for H = 8 can be shown to fail to give a solution. The one whole number solution then is

$$\begin{array}{r} 92836 \\ 12836 \\ \hline 105672 \end{array}$$

For P = 0 the correct solutions are

$$\begin{array}{r} 0.18479 \\ 0.08479 \\ \hline 0.26958 \end{array} \qquad\qquad \begin{array}{r} 0.28479 \\ 0.08479 \\ \hline 0.36958 \end{array}$$

Some alphametic exercises in addition (with solutions) are given below.

(a)
$$\begin{array}{r} \text{R} \\ \text{R} \\ +\text{R} \\ \hline \text{SR} \end{array}$$

(b)
$$\begin{array}{r} \text{TTT} \\ +\ \text{V} \\ \hline \text{WYYY} \end{array}$$

(c)
$$\begin{array}{rr} \text{SEND} & 9567 \\ +\text{MORE} & 1085 \\ \hline \text{MONEY} & 10652 \end{array}$$

(d)
$$\begin{array}{rr} \text{HAVE} & 1386 \\ \text{SOME} & 9076 \\ \hline \text{HONEY} & 10462 \end{array}$$

(e)
$$\begin{array}{rr} \text{FORTY} & 29786 \\ \text{TEN} & 850 \\ +\ \text{TEN} & 850 \\ \hline \text{SIXTY} & 31486 \end{array}$$

(f)
$$\begin{array}{rr} \text{THREE} & 79422 \\ +\ \text{FOUR} & 3104 \\ \hline \text{SEVEN} & 82526 \end{array}$$

(g)
$$\begin{array}{rr} \text{CROSS} & 96233 \\ +\text{ROADS} & 62513 \\ \hline \text{DANGER} & 158746 \end{array}$$

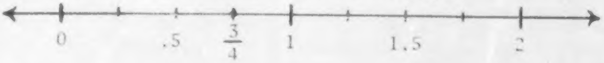
(h)
$$\begin{array}{rr} \text{ADAM} & 8384 \\ \text{AND} & 803 \\ \text{EVE} & 626 \\ \text{ON} & 50 \\ \text{A} & 8 \\ \hline \text{RAFT} & 9871 \end{array}$$

V. THE SET OF POSITIVE RATIONALS

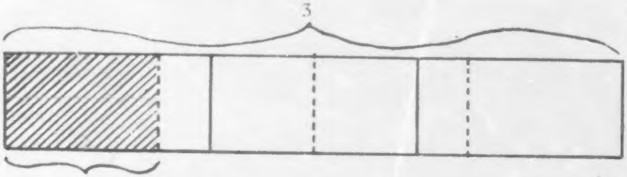
Although a *rational number* is defined as the quotient of two integers, $\frac{a}{b}$, where $b \neq 0$, it is not recommended that the student be given this definition at this level.

In the fraction $\frac{3}{4}$ the 4 is called the denominator (from the Latin "to tell by name") or divisor and tells by name the size of the parts we are talking about (here fourths) or we say it tells into how many *equal* parts the whole has been divided (4); the numerator (from the Latin "to count"), or dividend, tells how many of those equal parts we are concerned with (3). The fraction line indicates a division.

There are several interpretations of fractions that are important. *First* we can think of a fraction as a number having a position on the number line. Thus the fraction $\frac{3}{4}$ represents the number whose position is shown on the graph.

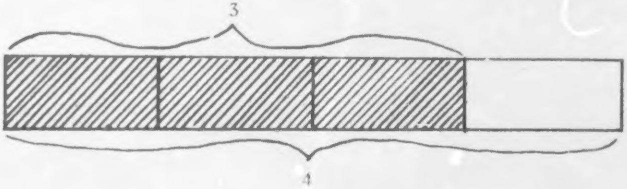


Second, the fraction $\frac{3}{4}$ represents the division 3:4 or $4\overline{)3}$ and can be shown as

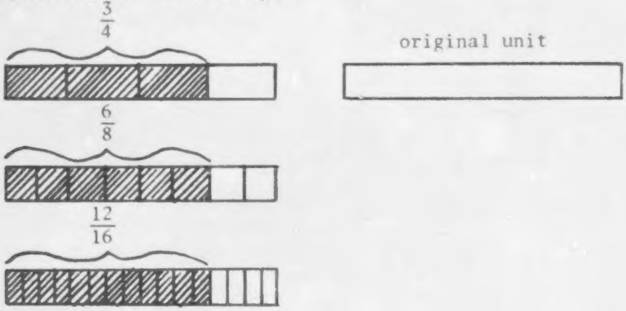


$\frac{1}{4}$ of 3 = $\frac{3}{4}$ of 1

which clearly points up the relationship $\frac{3}{4} \times 4 = 3$, as well as the relationship $\frac{1}{4}$ of 3 things is the same as $\frac{3}{4}$ of one thing. *Finally*, the fraction $\frac{3}{4}$ can also be thought of as 3 out of 4 things; the shaded portion of the diagram below $\frac{3}{4}$ of 4 things.



This idea of 3 out of 4, called a ratio, can be used to show the equivalence of fractions. Thus in the diagrams below we can see that $\frac{3}{4} = \frac{6}{8} = \frac{12}{16}$. Usually when we write a fraction like $\frac{3}{4}$ we mean three fourths of a whole thing.



It is important that we know that the whole thing is, because $\frac{3}{4}$ of 12 is not the same as $\frac{3}{4}$ of 8. It is rare that we use the rational numbers in practice except that they represent a fraction *of* something.

Subsets Of The Fractions

The word, fractions, refers either to the number or the numeral. Various subsets of the fractions are in common use. Some are:

(1) Equivalent or equal fractions - those fractions which represent the same number; for example, $\frac{1}{2}$, $\frac{12}{24}$, $\frac{4\frac{1}{2}}{9}$, $\frac{5}{10}$

($\frac{1}{2} = \frac{12}{24}$ because 1x24 = 2x12.)

(2) Simple or common fractions - fractions which have integral numerators and denominators: $\frac{2}{3}$, $\frac{8}{5}$, $\frac{125}{1000}$

(3) Unit fractions - simple fractions which have numerators of one: $\frac{1}{2}$, $\frac{1}{9}$, $\frac{1}{25}$, $\frac{1}{10}$

(4) Complex fractions - fractions with a fraction in the numerator or denominator or both: $\frac{2}{\frac{3}{5}}$, $\frac{\frac{1}{3}}{6}$, $\frac{2\frac{3}{5}}{\frac{4}{9}}$, $\frac{0.7}{0.5}$

(5) Similar or like fractions - fractions that have the same denominator: $\frac{2}{3}, \frac{1}{3}, \frac{8}{3}$

(6) Unlike fractions - fractions that have denominators which are not the same: $\frac{1}{3}, \frac{2}{5}, \frac{5}{6}$

(7) Proper fractions - fractions in which the numerator < denominator: $\frac{2}{3}, \frac{18}{25}, \frac{1}{100}$

(8) Improper fractions - fractions in which the numerator ≥ denominator: $\frac{3}{2}, \frac{5}{5}, \frac{7}{3}$

(9) Mixed number - the inferred addition of an integer and a fraction: thus the mixed number $3\frac{1}{2}$ means $3 + \frac{1}{2} = \frac{6}{2} + \frac{1}{2} = \frac{7}{2}$ as an improper fraction.

(10) Decimal fractions - are fractions whose denominators are powers of ten and are understood by the position of the decimal point; thus $0.2 = \frac{2}{10}$, $.05 = \frac{5}{100}$, $127.43 = (1 \text{x} 100) + (2 \text{x} 10) + (7 \text{x} 1) + (4 \text{x} \frac{1}{10}) + (3 \text{x} \frac{1}{100}) = 127 \frac{43}{100}$

(11) Per cent (%) - a common way of naming a fraction which has a denominator of 100; thus $5\% = \frac{5}{100}$, $84\% = \frac{84}{100}$, $3\frac{1}{2}\% = \frac{3\frac{1}{2}}{100}$, $100\% = \frac{100}{100}$, $250\% = \frac{250}{100}$

If the numerator of a fraction increases (decreases) as the denominator remains the same, the value of the fraction increases (decreases); thus $\frac{2}{7} < \frac{3}{7}$

If the denominator of a fraction increases (decreases) as the numerator remains the same, the value of the fraction decreases (increases); thus $\frac{3}{8} > \frac{3}{11}$

Simplifying Fractions

Concept: The greatest common divisor of two whole numbers is the largest whole number which divides both without a remainder.

In rewriting a fraction in simplest form (also use the expressions: *simplifying* or *reducing to lowest terms*), look for the largest whole number which will divide both the numerator and denominator of the fraction without a remainder. The greatest common divisor is the same as the greatest common factor. The greatest common divisor is not always evident, but students can find it by factoring the numerator and denominator. Divisibility rules should be reviewed and the use of them stressed at this point. The mathematically less able students can make effective use of the following method of finding the greatest common divisor.

When the denominator of a rational is the number 1, it is very common to drop the denominator and just write the numerator; for example, the rational $\frac{6}{1}$ is commonly written simply as 6, and the rational $\frac{1}{1}$ is commonly written as 1. When the numerator of a rational is zero, it is very common to drop the denominator and write the number simply as 0; for example $\frac{0}{4}$ is often written simply as 0.

The identity element for multiplication (one) plays an important role in expressing equivalent fractions in different forms, in simplifying complex fractions and in comparing the sizes of different fractions.

In simplifying fractions we can think of dividing numerator and denominator by a common factor or as factoring the fraction to obtain factors of one, as in the following:

a) $$\frac{8}{120} = \frac{2\cdot 2\cdot 2\cdot \overset{\text{assumed factor of 1, not 0}}{1}\cdot 1}{2\cdot 2\cdot 2\cdot 3\cdot 5} = 1\cdot 1\cdot 1\cdot \frac{1}{15} = \frac{1}{15}$$

b) $$\frac{90}{6} = \frac{2\cdot 3\cdot 3\cdot 5}{2\cdot 3\cdot 1\cdot 1} = \frac{15}{1} = 15$$

It might be worthwhile to write 1 as an additional factor when each factor of the numerator (denominator) has been matched with a like factor in the denominator (numerator) resulting in a fraction which has a 1 in the numerator or denominator.

Reduce the fraction $\frac{24}{36}$

both numerator and denominator are even numbers, so two is a divisor (or factor) of each: $\frac{\cancel{2}\cdot 12}{\cancel{2}\cdot 18} = \frac{12}{18}$

Again, numerator and denominator are even numbers, so two is a divisor (or factor) of each: $\frac{2\cdot 6}{2\cdot 9} = \frac{6}{9}$

Both numerator and denominator are divisible by three:

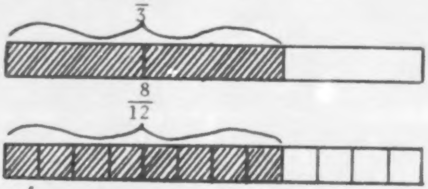
$$\frac{\cancel{3}\cdot 2}{\cancel{3}\cdot 3} = \frac{2}{3}$$

$$\therefore \quad \frac{24}{36} = \frac{2}{3}$$

$$\text{or } \frac{24}{36} = \frac{2\cdot 2\cdot 3\cdot 2}{2\cdot 2\cdot 3\cdot 3} = \frac{2}{3}$$

Increasing the Terms of a Fraction

If we multiply the numerator and denominator by the same number, as in $\frac{2 \times 4}{3 \times 4} = \frac{8}{12}$*, what have we done?* We have decreased the size of each part from thirds to twelfths, at the same time we have increased the number of parts from two to eight.



Notice also that $\frac{4}{4} = 1$, and we are really multiplying $\frac{2}{3}$ by one. Since any number multiplied by one is equal to the same number, the fraction $\frac{8}{12}$ must be equal to $\frac{2}{3}$.

$$\begin{array}{ccccc} \frac{2}{3} & \times & \frac{4}{4} & = & \frac{8}{12} \\ & & \downarrow & & \downarrow \\ \frac{2}{3} & \times & 1 & = & \frac{2}{3} \end{array}$$

To compare more easily the sizes of those fractions with which we are not familiar, we usually express them as equivalent fractions having a common denominator (or numerator). Per cents are used in this way because they represent fractions whose denominators are always 100. When we look for a common denominator, we are just looking for the common multiple of the denominators. To find the common denominator of $\frac{3}{4}$ and $\frac{2}{10}$ is merely to find among the natural numbers a common multiple of 4 and 10.

The multiples of 4 are: {4,8,12,16,20,24,28,32,36,40,...}
The multiples of 10 are: {10,20,30,40...}

The set of common multiples of 4 and 10 is {20,40,60,80...}

Notice that 4 and 10 have an infinite number of common multiples, the lowest of which, for the natural numbers, is 20. Now rewrite each fraction as an equivalent fraction with a denominator of 20 (or 40 or any common multiple). Using 20 has the advantage of dealing with fractions having smaller numerators and denominators.

To answer the question $\frac{3}{4} = \frac{?}{20}$, we may think of the form,

$\frac{3}{4} \times \frac{?}{?} = \frac{?}{20}$. We know we must multiply the fraction by 1 (the identity element) in some other form. *By what must we multiply 4 to get 20?* 5; therefore the 1 must be in the form $\frac{5}{5}$. Then $\frac{3}{4} \times \frac{5}{5} = \frac{15}{20}$ similarly, $\frac{2}{10} \times \frac{2}{2} = \frac{4}{20}$.

Comparing $\frac{4}{20}$ and $\frac{15}{20}$, we know that $\frac{15}{20} > \frac{4}{20}$ or $\frac{4}{20} < \frac{15}{20}$.

There are other ways of finding the lowest common multiple of two numbers without enumerating the multiples of each number. *Find the lowest common multiple (LCM) of 84 and 45.* First find the prime factorization of each number,

$84 = 2 \cdot 2 \cdot 3 \cdot 7$

$45 = 3 \cdot 3 \cdot 5$

Then form the product which contains each distinct factor that appears, the *greatest* number of times it occurs in any *one* number. Thus the L.C.M. of 84 and 45 is $2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 = 1260$.

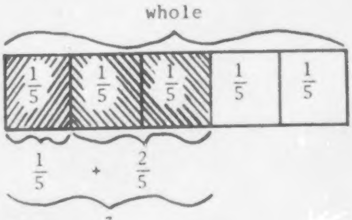
Operations with Fractions

Either the vertical or horizontal form for operating with fractions may be used - however, it is suggested that the horizontal form be emphasized.

Concept: Addition with like denominators.

Add: $\frac{1}{5} + \frac{2}{5}$

Answer: $\frac{1}{5} + \frac{2}{5} = \frac{1+2}{5} = \frac{3}{5}$

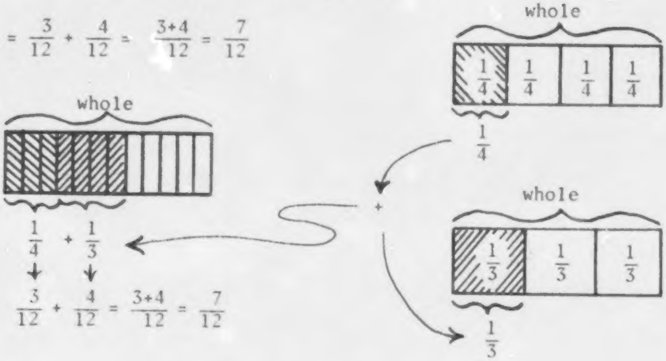


Concept: Addition with unlike denominators:

Add: $\frac{1}{4} + \frac{1}{3}$

Answer: A common denominator must be found.

$$\frac{1}{4} + \frac{1}{3} = \frac{3}{12} + \frac{4}{12} = \frac{3+4}{12} = \frac{7}{12}$$



While the lowest common denominator is the easiest to use, some students will be able to find *a* common denominator by using the following form which might not be the lowest common denominator but is acceptable.

This method follows the rule: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

i.e. a) $\frac{2}{3} + \frac{3}{5} = \frac{2(5)}{3(5)} + \frac{2(3)}{5(3)}$

$\frac{10}{15} + \frac{6}{15} = \frac{16}{15}$

b) $\frac{1}{4} + \frac{2}{6} = \frac{1(6)}{4(6)} + \frac{2(4)}{6(4)} = \frac{6}{24} + \frac{8}{24} = \frac{14}{24}$ which may then be simplified to $\frac{7}{12}$.

Concept: Addition of mixed numbers.

Vertical

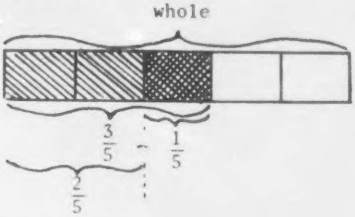
$$\begin{array}{r}
4\,\frac{1}{2} = 4\,\frac{3}{6} \\
+3\,\frac{2}{3} = 3\,\frac{4}{6} \\
\hline
= 7\,\frac{7}{6} = 8\,\frac{1}{6}.
\end{array}$$

Horizontal

$$\begin{aligned}
4\,\tfrac{1}{2} + 3\,\tfrac{2}{3} &= 4 + \tfrac{1}{2} + 3 + \tfrac{2}{3} \\
&= 4 + 3 + \tfrac{1}{2} + \tfrac{2}{3} \\
&= 4 + 3 + \tfrac{3}{6} + \tfrac{4}{6} \\
&= 7\,\tfrac{7}{6} = 8\,\tfrac{1}{6}
\end{aligned}$$

Concept: Subtraction with like denominators.

$$\frac{3}{5} - \frac{1}{5} = \frac{3-1}{5} = \frac{2}{5}$$



Concept: Subtract with unlike denominators.

As in addition, a common denominator must be found.

$$\frac{3}{4} - \frac{2}{6} = \frac{9-4}{12} = \frac{5}{12} \quad \text{or} \quad \frac{3}{4} - \frac{2}{6} = \frac{3(6)}{4(6)} - \frac{2(4)}{6(4)} = \frac{18}{24} - \frac{8}{24} = \frac{18-8}{24} = \frac{10}{24}$$

which simplifies to $\frac{5}{12}$

Concept: Subtraction of mixed numbers.

(A) Mixed number minus whole number

Vertical

$$\begin{array}{r} 8\frac{1}{3} \\ -\ 2 \\ \hline 6\frac{1}{3} \end{array}$$

Horizontal

$$8\frac{1}{3} - 2 = 8 + \frac{1}{3} - 2 = 8-2+\frac{1}{3} = 6+\frac{1}{3} = 6\frac{1}{3}$$

(B) Whole number minus mixed number

Vertical

$$\begin{array}{rcr} 9 & & 8\frac{6}{6} \\ -\ 3\frac{5}{6} & = & -\ 3\frac{5}{6} \\ \hline & & 5\frac{1}{6} \end{array}$$

Horizontal

$$9-3\frac{5}{6} = 9-3-\frac{5}{6} = 6-\frac{5}{6} = 5+\frac{6}{6} - \frac{5}{6} = 5+\frac{1}{6} = 5\frac{1}{6}$$

An alternate procedure for the vertical form may be used in B.

Rule: Add the same fraction to both numbers in order that the number subtracted is in the form of a whole number.

$$\begin{array}{rcrcr} 9 & & 9+\frac{1}{6} & = & 9\frac{1}{6} \\ -3\frac{5}{6} & & -(3\frac{5}{6}+\frac{1}{6}) & = & -4 \\ \hline & & & & 5\frac{1}{6} \end{array}$$

This involves the basic concept that adding the same quantity to two numbers leaves the difference between them unchanged.

(C) Mixed number minus mixed number where fractional part of subtrahend is less than the fractional part of minuend.

Vertical | Horizontal

(1)

$$\begin{array}{l} \quad 6\frac{3}{4} \\ -2\frac{1}{4} \\ \hline \quad 4\frac{2}{4} = 4\frac{1}{2} \end{array}$$

$$6\frac{3}{4} - 2\frac{1}{4} = 6 + \frac{3}{4} - 2 - \frac{1}{4} = 6-2 + \frac{3}{4} - \frac{1}{4} = 4 + \frac{2}{4} = 4\frac{1}{2}$$

(2)

$$\begin{array}{r} 5\frac{5}{6} = 5\frac{5}{6} \\ -3\frac{1}{3} = -3\frac{2}{6} \\ \hline 2\frac{3}{6} = 2\frac{1}{2} \end{array}$$

$$5\frac{5}{6} - 3\frac{1}{3} = 5 + \frac{5}{6} - 3 - \frac{1}{3} = 5-3 + \frac{5}{6} - \frac{1}{3} = 5-3 + \frac{5}{6} - \frac{2}{6} = 2 + \frac{3}{6} = 2\frac{3}{6} = 2\frac{1}{2}$$

(D) Mixed number minus mixed number where fractional part of subtrahend is greater than fractional part of minuend.

Vertical

(1)

$$\begin{array}{r} 4\frac{2}{7} = 4\frac{4}{14} = 3\frac{18}{14} \\ -2\frac{1}{2} = 2\frac{7}{14} = 2\frac{7}{14} \\ \hline 1\frac{11}{14} \end{array}$$

Horizontal

$$4\frac{2}{7} - 2\frac{1}{2} = 4 + \frac{2}{7} - 2 - \frac{1}{2} = 4 - 2 + \frac{2}{7} - \frac{1}{2} = 4 - 2 + \frac{4}{14} - \frac{7}{14} =$$
$$2 + \frac{4}{14} - \frac{7}{14} = 1 + \frac{14}{14} + \frac{4}{14} - \frac{7}{14} = 1 + \frac{18}{14} - \frac{7}{14} = 1 + \frac{11}{14} = 1\frac{11}{14}$$

An alternate procedure for the vertical form in D involves the addition of a fraction to both subtrahend and minuend, so that the subtrahend is in the form of a whole number.

$$\begin{array}{rrrr} 8\frac{1}{3} & 8\frac{1}{3} + \frac{1}{3} = & 8\frac{2}{3} = & 8\frac{2}{3} \\ -2\frac{2}{3} & -\left(2\frac{2}{3} + \frac{1}{3}\right) = & -\ 2\frac{3}{3} = & -3 \\ \hline & & & 5\frac{2}{3} \end{array}$$

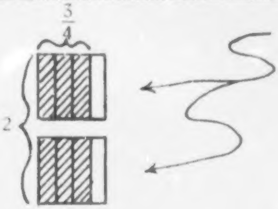
Concept: Multiplication of Rationals.

Any rational may be rewritten in the form $\frac{a}{b}$. Thus, $.75 = \frac{3}{4}$, $2 = \frac{2}{1}$, $3\,\frac{1}{3} = \frac{10}{3}$.

Multiplication in the set of rationals can be defined as $(\frac{a}{b})\cdot(\frac{c}{d}) = \frac{ac}{bd}$. The product of any two rationals is the rational whose numerator is the product of the numerators of the given rationals and whose denominator is the product of the denominators of the given rationals.

Example — Picture — Shaded Portion

(1) $2\mathrm{x}\frac{3}{4} = \frac{2}{1}\mathrm{x}\frac{3}{4} = \frac{6}{4} = \frac{3}{2}$

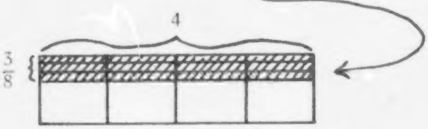


$6\mathrm{x}(\frac{1}{4}) = \frac{6}{4} = \frac{3}{2}$

Original Unit

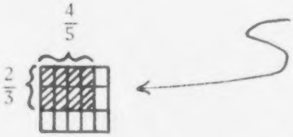
(2) $\frac{3}{8} \times 4 = \frac{3}{8} \times \frac{4}{1} = \frac{12}{8} = \frac{3}{2}$

$12\mathrm{x}\frac{1}{8} = \frac{12}{8} = \frac{3}{2}$



Original Unit

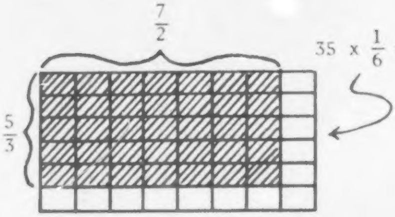
(3) $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$



$8\mathrm{x}\ \frac{1}{15} = \frac{8}{15}$

Original Unit

(4) $3\,\frac{1}{2} \times 1\,\frac{2}{3} =$
$\frac{7}{2} \times \frac{5}{3} = \frac{35}{6}$



$35 \times \frac{1}{6} = \frac{35}{6}$

Original Unit

If the product of any two numbers is one, each is called the *reciprocal* of the other. *Multiplicative inverse* is another name for reciprocal.

$$\frac{1}{2} \times \frac{2}{1} = \frac{2}{2} = 1$$

$$2\frac{1}{3} \times \frac{3}{7} = \frac{7}{3} \times \frac{3}{7} = \frac{21}{21} = 1$$

Concept: Division of Rationals.

There is usually a need for reteaching division of rationals. There are three ways to approach this topic:

(1) We realize that a factor x a factor = the product

$$a \times b = c$$

implies that $\dfrac{\text{the product}}{\text{one factor}} = \text{the other factor}$

$$\frac{c}{b} = a$$

or

$$\frac{c}{a} = b$$

Thus, $8 \times \square = 24$ implies that $\frac{24}{\square} = 8$ or $\frac{24}{8} = \square$

We have learned that for every $n (n \neq 0)$ there exists a number such that when multiplied by n gives the product 1. This number we call the multiplicative inverse of n or the reciprocal of n. Thus $n \cdot \frac{1}{n} = 1$; specifically

$4 \times \frac{1}{4} = 1$ where $\frac{1}{4}$ is the reciprocal of 4 (and 4 is the reciprocal of $\frac{1}{4}$)

$\frac{3}{4} \times \frac{4}{3} = 1$ where $\frac{4}{3}$ is the reciprocal of $\frac{3}{4}$ (and $\frac{3}{4}$ is the reciprocal of $\frac{4}{3}$)

In other words, any number times its reciprocal equals 1. Now we are ready to talk about the division of two rational numbers.

By the definition of division

$5 \div \frac{2}{3} = \boxed{}$ or $\frac{5}{\frac{2}{3}} = \boxed{}$ means

$\boxed{} \times \frac{2}{3} = 5.$ Since the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$

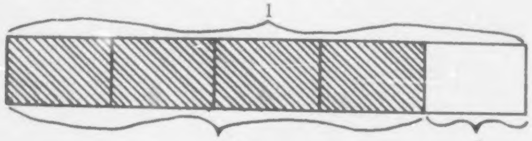
$\underbrace{\boxed{? \times \frac{3}{2}} \times \frac{2}{3}}_{1} = 5.$ Now, *what times one = 5?* 5.

$\boxed{5 \times \frac{3}{2}} \times \frac{2}{3} = 5.$ So the number we are looking for, the quotient of $\frac{5}{\frac{2}{3}}$ is in the box and is equal to $5 \times \frac{3}{2}$

$$\therefore 5 \div \frac{2}{3} = 5 \times \frac{3}{2} = \frac{15}{2}$$

The quotient of two rational numbers is the dividend multiplied by the reciprocal of the divisor. (The division written in this form, using the symbol ÷, has the number on the left as the dividend.)

(2) The first method involves division as the inverse of multiplication. This second way is showing that division is the process in which we find the number of times the divisor is contained in the dividend. Let us study the question, *how many* $\frac{4}{5}$*'s are contained in one whole thing?*

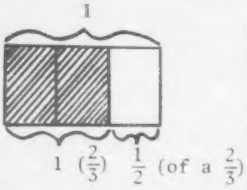


From the diagram we see that in one whole thing there is 1 (four-fifth) plus $\frac{1}{4}$ (of a four-fifth) or $1\frac{1}{4}$ four-fifths in 1 whole thing. Thus

$$\frac{1}{\frac{4}{5}} = 1\frac{1}{4} = \frac{5}{4}$$

(the $\frac{1}{4}$ refers to $\frac{1}{4}$ of the unit with which we are working, *a* $\frac{4}{5}$, *not* $\frac{1}{5}$ of 1.)

Notice that $\frac{5}{4}$ is the reciprocal of $\frac{4}{5}$. In other words a number is contained in 1, a number of times equal to its reciprocal.



How many $\frac{2}{3}$'s are contained in 1?

$$\frac{1}{\frac{2}{3}} = \frac{3}{2}\text{, because } \frac{2}{3}\cdot\frac{3}{2} = 1$$

How many 3's are contained in 1?

$$\frac{1}{3} = \frac{1}{3} \text{ because } 3\cdot\frac{1}{3} = 1$$

How many $4\,\frac{2}{3}$'s or $\frac{14}{3}$'s are contained in 1?

$$\frac{1}{\frac{14}{3}} = \frac{3}{14} \text{ because } \frac{14}{3} \times \frac{3}{14} = 1.$$

Now we ask the question $5 \div \frac{2}{3} = ?$ or $\frac{5}{\frac{2}{3}} = ?$

In words, *how many $\frac{2}{3}$'s are contained in 5?* In 1 whole thing there are $\frac{3}{2}$ of them, so in 5 there are $5\times\frac{3}{2}$

$$\therefore 5\div\frac{2}{3} = 5\times\frac{3}{2} = \frac{15}{2}\text{, as in method (1).}$$

Again the quotient of two rational numbers is the dividend multiplied by the reciprocal of the divisor.

(3) The third method of showing the division of two fractions, like the method of simplifying a complex fraction, involves using the identity element to increase the terms of the fraction. Thus $\frac{2}{3} \div \frac{4}{5}$ can be written

$$\frac{\frac{2}{3}}{\frac{4}{5}}$$

If we now multiply this complex fraction by the identity element for multiplication, 1, in the form $\frac{15}{15}$ or $\frac{\frac{15}{1}}{\frac{15}{1}}$

(where 15 is the LCM of the denominators of the two fractions), we obtain

$$\frac{\frac{2}{3} \times \frac{15}{1}}{\frac{4}{5} \times \frac{15}{1}} = \frac{10}{12} = \frac{5}{6} .$$

We could also have multiplied by 1 in the form

$$\frac{\frac{5}{4}}{\frac{5}{4}}$$ where $\frac{5}{4}$ is the reciprocal of $\frac{4}{5}$

Properties of Positive Rationals

Under addition	Under multiplication
Closure: $a+b$ = a rational number	Closure: $a \times b$ = a rational number
Commutative property: $a+b = b+a$	Commutative property: $a \cdot b = b \cdot a$
Associative property: $a+(b+c) = (a+b)+c$	Associative property: $a(bc) = (ab)c$
Identity element: There is an element z in the set such that $a+z = a$ (z = zero for the rational numbers)	Identity element: There is an element u in the set such that $a \cdot u = a$ (u = 1 for the rational numbers)
Additive inverse: For every element a there is an element *a such that $a+{*a} = {*a}+a$ (*a for the rational numbers is -a)	Multiplication inverse: For every element a there is an element a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = u$ (a^{-1} for the rational numbers is $\frac{1}{a}$)

Distributive Property
$a(b+c) = (a \cdot b)+(a \cdot c)$ and
$(b+c) \cdot a = (b \cdot a) + (c \cdot a)$

Another property of the positive rationals is the density property.

Concept: The density property states that between any two rational numbers there is another rational number.

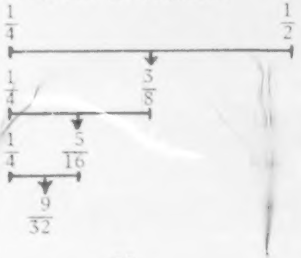
Given any two different rationals, it is always possible to determine a rational which is greater than one of them and less than the other rational. For example to determine a rational that is greater than $\frac{5}{11}$ and less than $\frac{6}{11}$, replace each by an equivalent rational with greater denominator, thus $\frac{10}{22}$ and $\frac{12}{22}$ respectively. The rational $\frac{11}{22}$ is greater than $\frac{10}{22}$ and less than $\frac{12}{22}$. This means that $\frac{11}{22}$ is greater than $\frac{5}{11}$ and less than $\frac{6}{11}$.

To determine a rational that is between any two given rationals, determine rationals equivalent to each of the given rationals such that the replacements have the same denominator, that is, they are in the form $\frac{a}{b}$ and $\frac{c}{b}$. If d is between a and c, then $\frac{d}{b}$ is between the given rationals. If a and c are consecutive, multiply the numerator and denominator of each expression by 2 so that they are in the form $\frac{2a}{2b}$ and $\frac{2c}{2b}$. The numerators are now even integers. There exists an odd integer that is between the two even integers. When this odd integer is placed over the common denominator, 2b, the expression is formed for the rational which is half way between the two given rationals. Between any two rationals, therefore, there exists another rational. In fact, between any two rationals there exists an infinite number of rationals.

Given any rational, there is no way of determining the next greater or the next smaller rational. If any rational is suggested as the next greater or the next smaller, another rational can be proven to exist between that rational and the given rational.

If there is no way of determining the next greater or the next smaller rational, then counting cannot be performed in the set of rationals in the same manner as it was in the set of natural numbers and the set of integers. However, as was previously shown, certain subsets of the rationals, that is rationals with the same denominator, can be counted.

Problem: Picture the insertion of three rationals between $\frac{1}{4}$ and $\frac{1}{2}$.

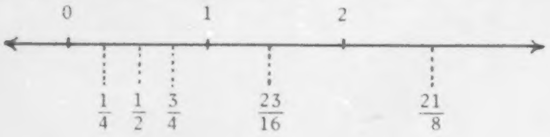


Answer:

$\frac{3}{8}$, $\frac{5}{16}$, $\frac{9}{32}$,

(There are many others).

Concept: Rational numbers may now be added to the number line.



Concept: Simplyfing complex fractions is analogous to the division of rationals.

We may use the multiplicative inverse of the denominator to multiply both numerator and denominator.

a) $$\frac{\frac{1}{2}}{2} = \frac{\frac{1}{2} \cdot \frac{1}{2}}{2 \cdot \frac{1}{2}} = \frac{\frac{1}{4}}{1} = \frac{1}{4}$$

b) $$\frac{6}{\frac{2}{3}} = \frac{6 \cdot \frac{3}{2}}{\frac{2}{3} \cdot \frac{3}{2}} = \frac{\frac{18}{2}}{\frac{6}{6}} = \frac{9}{1} = 9$$

c) $$\frac{\frac{1}{2}}{\frac{1}{4}} = \frac{\frac{1}{2} \cdot \frac{4}{1}}{\frac{1}{4} \cdot \frac{4}{1}} = \frac{\frac{4}{2}}{\frac{4}{4}} = \frac{2}{1} = 2$$

Concept: Base ten notation can be extended to write fractions in decimal form.

In any decimal numeral each place is $\frac{1}{10}$ the value of the place to its left and 10 times the value of the place to its right. In discussions of the decimal fractions, place values, to the right of the decimal point can be presented as powers of $\frac{1}{10}$ in addition to naming them. Decimal fractions can now be written in three forms

a)	.623	positional
b)	6x.1+2x.01+3x.001	polynomial or expanded form
c)	$6x\frac{1}{10}+2x\frac{1}{10^2}+3x\frac{1}{10^3}$	exponential form

Concept: Writing a common fraction as a decimal fraction.

$\frac{4}{5}$ means 4÷5 or $5\overline{)\,4\,}$. Since 4=4.0=4.00=4.000 and 5 will not divide 4 an integral number of times, we use 4.0 as 40 tenths divided by 5, the quotient being 8 tenths.

```
    0.8
5)  4.0
```

There is no question then as to where the decimal point is placed. If 4.05 is divided by 5, we can think 405 hundredths divided by 5, which results in 80 hundredths, then 1 hundredth, or 0.81 as the quotient. In changing $\frac{4}{5}$ to a decimal fraction we notice that the decimal terminates, that is we get a remainder of zero (we could also think of it as repeating a zero.) Suppose we change $\frac{2}{7}$ to a decimal

```
  .01
  .80
5)4.05
  4.00
   .05
   .05
```

```
  .285714285714
7)2.0000
 -1 4
    60
   -56
    40
   -35
     50
    -49
      10
      -7
      30
     -28
       20
        .
        .
        .
```

The decimal equivalent of $\frac{2}{7}$ is what we call an infinite repeating or periodic decimal because it never terminates and the digits 285714 repeat. We say $\frac{2}{7} = 0.\overline{285714}...$

We put a line over 285714 to show that these digits repeat as a group. The period or number of repeating digits is 6.

$\frac{1}{3} = .\overline{3}$ ... although we usually write $\frac{1}{3} = 0.33\frac{1}{3}$ expressed as hundredths

$\frac{1}{9} = .1111... = .\overline{1}$ ...

$\frac{2}{9} = .222... = .\overline{2}$ ...

$\frac{3}{9} = .22... = .\overline{3}$ ..., the ninths being an interesting set of fractions. The sevenths are also very interesting because the decimal equivalent is the same 6 digits repeated but in a different order.

$\frac{1}{7} = 0.\overline{142857}\ \ldots;\ \frac{2}{7} = 0.\overline{285714}\ \ldots;\ \frac{3}{7} = 0.\overline{428571}\ \ldots;\ \ldots$

Concept: Any common fraction or rational number can be written as either a terminating or infinitely repeating decimal.

Express $\frac{2}{3}$ *as tenths:* $0.6\frac{2}{3}$ (6 and $\frac{2}{3}$ tenths);

hundredths: $0.66\frac{2}{3}$ (66 and $\frac{2}{3}$ hundredths);

thousandths: $0.666\frac{2}{3}$ (666 and $\frac{2}{3}$ thousandths)

Express $\frac{2}{3}$ *as a decimal rounded to the nearest*

tenth: 0.7; *hundredth:* 0.67; *thousandth:* 0.667

Whenever a division is asked for, the form in which the answer is to be shown should be clear; that is, whether to round off to a certain place or to express the answer as hundredths or to leave the quotient as a common fraction.

Concept: Any terminating decimal may be easily expressed as a common fraction by the use of the place values and simplifying.

Thus we can read, write and reduce as follows:

$$0.28 = \frac{28}{100} = \frac{7}{25}$$

$$0.12\frac{1}{2} = \frac{12\frac{1}{2}}{100} \cdot \frac{2}{2} = \frac{25}{200} = \frac{1}{8}$$

$$0.03\frac{3}{4} = \frac{3\frac{3}{4}}{100} = \frac{3\frac{3}{4} \cdot 4}{100 \cdot 4} = \frac{15}{400} = \frac{3}{80}$$

$$0.6\frac{2}{3} = \frac{6\frac{2}{3}}{10} = \frac{6\frac{2}{3} \cdot 3}{10 \cdot 3} = \frac{20}{30} = \frac{2}{3}$$

$$0.052 = \frac{52}{1000} = \frac{13}{250}$$

Concept: Writing a repeating decimal as a common fraction.

We see that we can write a terminating decimal as a common fraction, but *can we write a repeating decimal as a common fraction?* Yes, very simply. Suppose we want to find a common fraction equivalent to $0.434\overline{343}\ldots$

$$\begin{array}{lcl}
\text{since} \quad x & = & 0.434\overline{343}\ldots \\
100 \quad x & = & 43.434\overline{343}\ldots \\
- \quad x & = & -0.434\overline{343}\ldots \quad \text{subtracting x} \\
\hline
99 \quad x & = & 43 \\
x & = & \dfrac{43}{99}
\end{array}$$

Find the common fraction equivalent to $0.25\overline{55}\ldots$.

$$\begin{array}{lcl}
\text{Let} \quad x & = & 0.25\overline{55}\ldots \text{ then} \\
10 \quad x & = & 2.5\overline{55}\ldots \\
\text{and} \quad 100 \quad x & = & 25.5\overline{55}\ldots \\
-\ 10 \quad x & = & 2.5\overline{55}\ldots \\
\hline
90 \quad x & = & 23 \\
x & = & \dfrac{23}{90}
\end{array}$$

Concept: Rounding decimal fractions.

When rounding decimals to a particular place, that place must contain the last digit of the numeral which is written. Any additional zeroes written would imply a rounding off to the last place in which a zero appears. Neither may a zero be dropped if it is in the place rounded to.

Round 3.236 to the nearest hundredth

Answer: 3.24 is correct
3.240 is incorrect, for the zero implies a rounding to the nearest thousandth.

Round 4.397 to the nearest hundredth.

Answer: 4.40 is correct
4.400 is incorrect for the zero implies a rounding off to thousandths
4.4 is incorrect for the last digit, 4, in the tenths place implies a rounding off to tenths.

Concept: Comparison of size of decimal fractions.

Start with the tenths place and work to the right comparing the corresponding digits in each place. If the place values present in the numbers being compared are not in a 1:1 correspondence, zeros may be added to right of the last digit written.

Which is the larger, 0.1132 or 0.1136?

0 . 1 1 3 2
↑ ↑ ↑ ↑ ↑ ↑
↓ ↓ ↓ ↓ ↓ ↓
0 . 1 1 3 6

We must inspect digits in the ten thousandths place before a decision may be made.

Answer: 0.1136 is larger.

Which is the smaller, 0.6 or 0.61?

0 . 6 0
↕ ↕ ↕ ↕
0 . 6 1

Some students may need to place a zero in the hundredths place in order to compare.

Answer: 0.6 is smaller.

In order to compare sizes of common fractions and decimal fractions - the form of one type fraction must be changed. Do not insist that only changing the common fraction to decimal form is correct - it is (probably) the easiest - but changing the decimal fraction to common fraction form is just as correct.

Concept: Addition of decimal fractions.

In the vertical form like place values must be under one another.

Only digits in like place values, may be added.

Vertical

$$\begin{array}{r} .3 \\ +.04 \\ \hline .34 \end{array}$$

or

Horizontal

$$\frac{3}{10} + \frac{4}{100} = \frac{30}{100} + \frac{4}{100} = \frac{34}{100} = .34$$

It may be necessary for some students to add zeroes as place holders.

i.e.

$$\begin{array}{r} .30 \\ +\ .04 \\ \hline .34 \end{array}$$

Concept: Subtraction of decimal fractions.

In the vertical form the place values must be under one another. Only digits in like place values may be subtracted.

Vertical

$$\begin{array}{r} 9.64 \\ -5.05 \\ \hline 4.59 \end{array}$$

or

Horizontal

$$9+\frac{64}{100} - 5 - \frac{5}{100} = 9-5+\frac{64}{100} - \frac{5}{100} =$$

$$4 + \frac{59}{100} = 4.59$$

The rules for placing the decimal point in the multiplication and division of decimals may be given more meaning by working with the corresponding common fractions.

Multiplication: $.5 \times .17 =$ or $\frac{5}{10} \times \frac{17}{100} = \frac{85}{1000} = .085$

Division: $.6\overline{).42}$ $\begin{array}{r} .7 \\ 6\overline{)\,4.2} \end{array}$ or $\frac{.42}{.6} = \frac{.42\text{x}100}{.6\text{x}100} = \frac{42}{60} = \frac{7}{10} = .7$

VI. THE SET OF INTEGERS

Concept: An integer is any element of the set {..., *-3,-2,-1,0,1,2, 3,...*}

The numbers to the left of zero are called the negative integers. The positive integers are to the right of zero. These two sets of numbers and zero are the integers.

-5
-4
-3
-2
-1
0
1
2
3
4
5

Concept: The integers may be shown on the number line.

The arrows at the ends of the number line indicate continuation in both directions. The sign which shows that a number is positive is a small "+", which is written in front of the number, i.e. $^{+}2$; the sign which shows that a number is negative is a small "-", which is written in front of the number, i.e. $^{-}4$. Parentheses may be used to enclose signed numbers to avoid confusion with operation signs i.e. $(^{+}2)$, $(^{-}4)$.

In some texts, the signs "+" and "-" are written in a raised position as shown:

$^{-}5$ or $^{+}3$

This method of sign notation precludes the necessity for parentheses.

$^{-}5 + {}^{+}3$

In this outline, the signs will not be raised and parentheses will be used where needed.

In common usage, the positive sign is not written, but in beginning work with signed numbers, teachers would do well to insist that it be used, at least until the students do not confuse it with the addition sign.

Concept: Zero is neither positive nor negative.

When the integers are arranged on a horizontal number line as previously shown, the direction from zero toward the right is called the positive direction and the direction from zero toward the left is called the negative direction.

Concept: When the integers are arranged in the manner previously shown, any integer is greater than any integer to its left.

This means that 0 is greater than -16 and also $^{+}3$ is greater than -25.

In symbols: $0 > -16$, and $+3 > -16$

Counting can be carried on indefinitely in the positive direction and also indefinitely in the negative direction. There is no greatest integer, and there is no least integer.

Addition

In the set of natural numbers, addition was defined as a counting operation. This same concept applies to the set of integers, but the definition is now expanded.

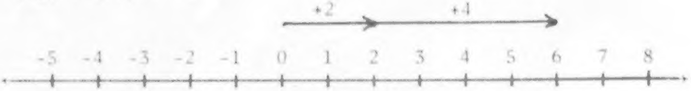
Concept: The definition of addition in the set of integers involves three cases: (1) $a + (+b)$, (2) $a + (-b)$, (3) $a + 0$ where a is any integer, $+b$ is any positive integer, $-b$ is any negative integer, and 0 is the integer zero.

(1) The sum $a + (+b)$ is obtained by counting "b" consecutive numbers from a in the positive direction.
(2) The sum $a + (-b)$ is obtained by counting "b" consecutive numbers from a in the negative direction.
(3) The sum $a + 0$ is always the number a.

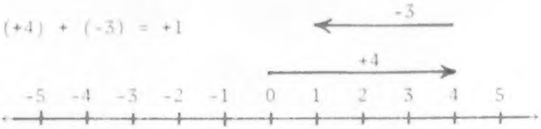
The use of a number line can be very helpful in introducing the concept of addition in the integers, and can be used by the pupils to advantage in performing exercises in such addition.

Examples

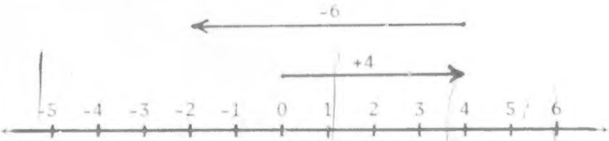
(1) $(+2) + (+4) = +6$



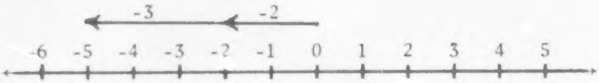
(2) $(+4) + (-3) = +1$



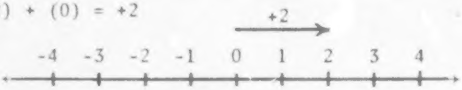
(3) $(+4) + (-6) = -2$



(4) (-2) + (-3) = -5



(5) (+2) + (0) = +2



As the pupil continues to perform exercises, he probably will soon realize that performing addition by counting is impractical where larger numbers are involved. Methods must be developed to perform addition in some manner other than counting.

The pupil should be led to realize that the sum of two positive integers is always a positive integer. In adding two positive integers, the counting begins at a positive number and is carried out in the positive direction. The sum must therefore always be a positive number. The sum of two negative numbers is always a negative number. In adding two negative numbers, the counting begins at a negative number and is carried out in the negative direction. The sum must therefore always be a negative number.

In adding two integers of opposite sign, counting begins at the first number and is carried out from that number toward zero. If the second number is small enough so that the counting does not go to zero, the sum will have the same sign as the first number. If the second number is such that the counting goes on past zero, the sum will have the sign opposite to that of the first number.

The rules that are used in the addition of integers should *not* be given to the pupils, but they should be encouraged to develop the necessary rules themselves from what they learn by solving exercises.

Subtraction

Concept: As in the set of positive integers, subtraction is related to a corresponding addition sentence.

Thus, 9-5 = □ may be read as 5 and what makes 9, or in symbols 5 + □ = 9. Subtraction in the set of integers may be introduced, similarly, through an equivalent addition sentence. Thus, a - (+b) = □ may be written as (+b) + □ = a and a - (-b) = □ may be written as (-b) + □ = a

Examples

1. (+6) - (+8) = ☐

This may be written as

(+8) + ☐ = (+6)

To go from +8 to +6, two numbers must be counted in the negative direction.

Thus ☐ = -2

2. (+6) - (-3) = ☐

Thus may be written as

(-3) + ☐ = (+6)

To go from -3 to +6, nine numbers must be counted in the positive direction.

Thus ☐ = +9

Concept: The definition of subtraction in the set of integers involves three cases: (1) $a - (+b)$, (2) $a - (-b)$, (3) $a - 0$, where a is any integer, $+b$ is any positive integer, $-b$ is any negative integer and 0 is the integer zero.

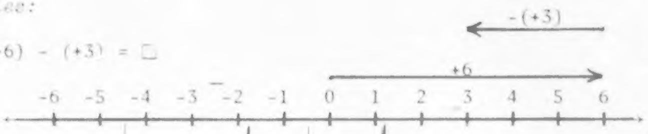
(1) the difference $a - (+b)$ is obtained by starting at a and counting b consecutive units in the negative direction.

(2) the difference $a - (-b)$ is obtained by starting at a and counting b consecutive units in the positive direction.

(3) the difference $a - 0$ is always a.

Examples:

1. (+6) - (+3) = ☐

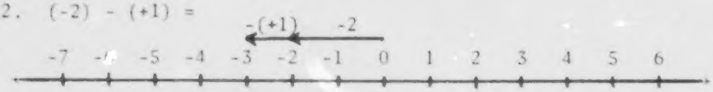


+ 6 + (-3) = +3

or

+ 6 - (+3) = +3

2. (-2) - (+1) =

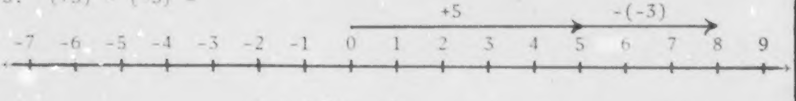


-2 + (-1) = -3

or

-2 - (+1) = -3

3. (+5) - (-3) =

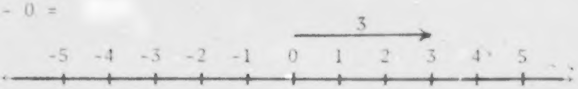


+5 + (+3) = +8

or

(+5) - (-3) = +8

4. 3 - 0 =



3 - 0 = 3

The subtraction a - (+b), where a is any integer and +b is any positive integer, may be performed by calculating the sum a + (-b).

The subtraction a - (-b), where a is any integer and -b is any negative integer, may be performed by calculating the sum a + (+b).

Subtraction in the integers may therefore be performed by adding the additive inverse of the subtrahend to the minuend. The subtraction -6 - (-9) may be obtained by performing the addition -6 + (-9), which is -15. The subtraction -6 - (-9) is equivalent to the addition -6 + (+9), which is 3.

The rules that are used in the subtraction of integers should not be given to the pupils, but they should be encouraged to formulate the necessary rules themselves from what they learn by solving exercises.

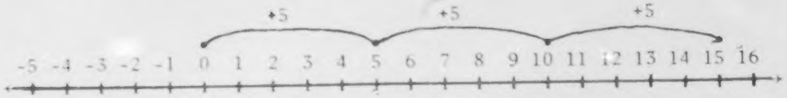
Multiplication

The definition of multiplication in the set of integers involv the following four cases: (1) (+a)(+b); (2) (+a)(-b),

(3) (-a)(+b); (4) (-a)(-b), where +a and +b are any positive integers, and -a and -b are any negative integers.

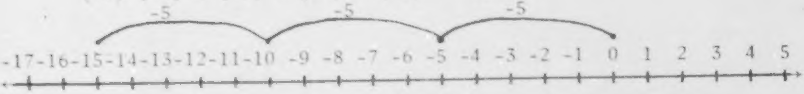
(1) *Concept: The product of +a and +b is obtained by finding the sum of "a" addends each of which is the number +b.*

(+3)•(+5) = (+5) + (+5) + (+5) = +15



(2) *Concept: The product of +a and -b is obtained by finding the sum of "a" addends, each of which is -b.*

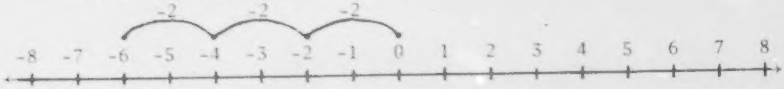
(+3)•(-5) = (-5) + (-5) + (-5) = -15



(3) The commutative principle for multiplication is an axiom in the set of integers. Thus, (-a)(+b) = (+b)(-a).

Concept: The product of (-a)(+b) can be obtained by finding the sum of "b" addends each of which is -a.

(-2)•(+3) = (+3)•(-2) = -6



(4) The last remaining portion of the definition of multiplication pertains to the product of two negative integers. The difficulty in presenting this portion of the definition is that it cannot be presented in terms of addition. Some other method must be used to present this concept of multiplication.

A method of determining the product of two negative integers is based on two principles, the principle that the product of zero and any integer is zero, and the principle that multiplication is distributive over addition in the set of integers.

Demonstrate the product in the general case, $(-a)(-b)$, where $-a$ and $-b$ are any negative integers.

$$\begin{aligned}(-a)(0) &= 0\\ (-a)[(-b) + (+b)] &= 0\\ (-a)(-b) + (-a)(+b) &= 0\\ (-a)(-b) + (-ab) &= 0\end{aligned}$$

The only integer that will result in a sum of zero when added to $-ab$ is the integer $+ab$.

Concept: Therefore, the product of $-a$ and $-b$ must be the integer $+ab$.

The problem of determining the product of any two negative integers such as -3 and -4 can be solved in the following manner:

$(-3)(0) = 0.$	The product of zero and any integer is zero.
$(-3)[(-4) + (+4)] = 0.$	The sum of any integer and its inverse is zero.
$(-3)(-4) + (-3)(+4) = 0.$	Multiplication is distributive over addition.
$(-3)(-4) + (-12) = 0.$	The product of -3 and +4 is -12.

The only integer that can be added to -12 to give a sum of zero is +12. Therefore, the product of -3 and -4 must be the integer +12.

Another method of presenting the concept of the multiplication of two negative integers is to observe the pattern of the products of a series of multiplications such as the following:

$$\begin{aligned}(+3)(-1) &= -3\\ (+2)(-1) &= -2\\ (+1)(-1) &= -1\\ (0)(-1) &= 0\\ (-1)(-1) &= ?\end{aligned}$$

If the same number pattern is to continue, the product of -1 and -1 would be the integer +1. The product of two negatives would be a positive number: $(-a)(-b) = +ab$.

There are various other procedures which may be used to clarify the concept of multiplication of integers and give the pupils a feeling that the multiplication rules make sense. Teachers are advised to consult textbooks and supplemental materials for explanations and uses of them.

Division

Concept: If the quotient is an integer, division in the integers is defined as the inverse operation of multiplication.

Division by zero is undefined and not permitted. Any division, $a : b = c$, that has a quotient in the integers, may be performed by carrying out the equivalent multiplication: $(c)(b) = a$.

If the quotient $(+a) : (+b) = \square$ is an integer, it may be obtained by calculating what integer multiplied by $+b$ will result in a product $+a$. That is, $(\square)\cdot(+b) = +a$. The quotient which replaces $\square$ cannot be a negative integer because the product of a negative integer and positive integer is a negative integer. The quotient cannot be zero because the product of any integer and zero is zero. Therefore, if the quotient of $(+a) : (+b)$ is an integer, it must be a positive integer. The quotient of $(+36) : (+9)$ may be obtained in the following manner:

$$(+36) : (+9) = \square$$
$$(\square)\cdot(+9) = +36$$
$$\square = +4$$

If the quotient in $(-a) : (-b) = \square$ is an integer, it may be obtained by calculating what integer multiplied by $-b$ will result in a product $-a$; that is, $(\square)\cdot(-b) = -a$. The quotient represented by $\square$ can only be a positive integer because only a positive integer can be multiplied by $-b$ to give a product that is a negative integer. The quotient of $-36 : -9$ may be obtained in the following manner:

$$(-36) : (-9) = \square$$
$$(\square)\cdot(-9) = -36$$
$$\square = +4$$

If the quotients are integers, the division $(+a) : (-b) = \square$ is written as the equivalent multiplication $(\square)\cdot(-b) = +a$ and the division $(-a) : (+b) = \square$ is written as the equivalent multiplication $(\square)\cdot(+b) = -a$. In each case, the quotient represented by $\square$ must be a negative integer, because the product of two integers with like (unlike) signs is a positive (negative) integer.

Concept: If the quotient of two integers with like signs is an integer, it is a positive integer. If the quotient of two integers with unlike signs is an integer, it is a negative integer.

The division $0 : a = \square$, if 0 is the integer zero and a is any integer, may be written as the equivalent multiplication $(\square)(a) = 0$. The quotient which replaces $\square$ must be the integer zero because if the product of two integers is zero, one of the given integers must be zero. The integer a cannot be zero as division by zero is not permitted. The quotient $0 \div a$ must therefore be zero.

In both multiplication and division, do not give the rules to the pupils, but let them formulate and verbalize them after they have had practice solving exercises.

Summary of the Properties of the Integers. The set of integers has the following properties:

- *(a) Closure under addition, subtraction, and multiplication*
- *(b) Commutative property for addition and multiplication*
- *(c) Associative property for addition and multiplication*
- *(d) Multiplication is distributive over addition and subtraction*
- *(e) Identity element for addition and multiplication*
- *(f) Inverse element for each integer under addition*
- *(g) There exists no greatest and no least integer.*

The integers do not have closure for division. The integers do not have inverse elements for each integer under multiplication. For every integer, a, there does not exist an integer, b, such that their product is +1.

The order of operations with integers is also multiplication and division first in the order in which they appear from left to right, and then addition and subtraction in the order in which they appear from left to right.

Absolute Value

A convenient way of telling how far from zero a number is on a number line is to use absolute value. The conventional symbolization, $|x|$ may be used and is read: "the absolute value of x." Since -6 and +6 are equidistant from zero, then $|-6| = |+6| = 6$, which means -6 and +6 are both 6 units from zero.

Concept: The $|x|$ is the number itself if x is positive or zero, and is $-x$ the additive inverse of x, if x is the negative number.

The absolute value of a number then, may never be negative.

VII. THE COMPLETE SET OF RATIONALS

Although this unit is planned for the eighth grade, parts of certain topics, such as measurement and square root, may be introduced in the seventh grade. Much depends on the readiness and ability of the pupils in a particular class.

Concept: The quotient of any two integers a and b ($b \neq 0$) can be expressed as the rational number $\frac{a}{b}$.

Like the integers, the rationals are either positive $+\frac{a}{b}$, negative $-\frac{a}{b}$ or if $a = 0$, then $\frac{a}{b} = 0$.

Concept: The quotient of two integers that have like signs is a positive number.

Therefore, the quotient $\frac{+a}{+b}$ must be the positive rational $+\frac{a}{b}$. The quotient $\frac{-a}{-b}$ must be the positive rational $+\frac{a}{b}$. Thus $\frac{+a}{+b}$, $\frac{-a}{-b}$, and $+\frac{a}{b}$ are three different ways of expressing the same rational number, and it follows that $\frac{+a}{+b} = \frac{-a}{-b} = +\frac{a}{b}$.

Concept: The quotient of two integers that have unlike signs is a negative number.

Therefore, the quotient $\frac{-a}{+b}$ must be the negative rational $-\frac{a}{b}$ The quotient $\frac{+a}{-b}$ must also be the rational $-\frac{a}{b}$. Thus $\frac{-a}{+b}$, $\frac{+a}{-b}$, and $-\frac{a}{b}$ are three different expressions for the same rational number and it follows that $\frac{-a}{+b} = \frac{+a}{-b} = -\frac{a}{b}$.

Just as it is common practice to drop the "+" sign from in front of the positive integers, so it is also common practice to drop the "+" sign from in front of the positive rationals and $+\frac{a}{b}$ may be written as $\frac{a}{b}$.

In the set of integers, division by zero is not permitted. Therefore, in the set of rationals, there does not exist any number whose denominator is zero.

Concept: The set of rationals has the property that for every rational there is an infinite number of different names for that rational number.

If two different expressions are names for the same number they are said to be equivalent.

Two rational expressions, $\frac{a}{b}$ and $\frac{c}{d}$, are equivalent if and only if ad = cb. Two rational expressions are expressions for the same number if and only if the product of the numerator of the first and denominator of the second equals the product of the denominator of the first and the numerator of the second. This is frequently called the cross product test. Applying this principle, $\frac{+a}{+b} = \frac{-a}{-b}$ and $\frac{+a}{-b} = \frac{-a}{+b}$. Through further application of this principle, it can be shown in still another way that a positive rational and a negative rational cannot be equal.

$\frac{+a}{-b} \neq \frac{+a}{+b}$, since $ab \neq - ab$

$\frac{+a}{-b} \neq \frac{-a}{-b}$, since $- ab \neq + ab$

Any two rational expressions whose numerators are zero are equivalent. Using the above principle $\frac{0}{a} = \frac{0}{b}$, since $0 \times b = 0 \times a$
$0 = 0$.

When arranging rationals on a number line, the positive rationals are written to the right of zero and the negative rationals to the left of zero. The rationals which have numerators equal to zero and denominators unequal to zero are all equal to zero.

Any positive rational is greater than any negative rational or the rational equivalent to zero. The rational equivalent to zero is greater than any negative rational.

Rationals with the same denominators, which make up part of the rationals, may be arranged in order as shown in this example of rationals with denominator of 3.

$$\ldots -\frac{5}{3},\ -\frac{4}{3},\ -\frac{3}{3},\ -\frac{2}{3},\ -\frac{1}{3},\ \frac{0}{3},\ \frac{1}{3},\ \frac{2}{3},\ \frac{3}{3},\ \frac{4}{3},\ \frac{5}{3},\ \ldots$$

When rationals with the same denominator are arranged in order as shown, any given rational in this list is greater than any rational to its left.

Addition of Rationals

Concept: Rationals may be added when the denominators are changed to a common form, then the numerator of the sum is always the sum of numerators of the rationals being combined.

a) $\frac{3}{7} + \frac{-1}{7} = \frac{3 + (-1)}{7} = \frac{2}{7}$

b) $\frac{-5}{6} + \frac{1}{3} = \frac{-5 + 2}{6} = \frac{-3}{6}$ or $-\frac{1}{2}$

c) $\frac{-2}{5} + \frac{-1}{5} = \frac{-2 + (-1)}{5} = \frac{-3}{5}$ or $-\frac{3}{5}$

d) $\frac{-1}{4} + \frac{-1}{3} = \frac{3(-1) + 4(-1)}{12} = \frac{-3 + (-4)}{12} = \frac{-7}{12}$ or $-\frac{7}{12}$

Subtraction of Rationals

Concept: Subtraction of rationals may be carried out as the equivalent addition, using subtraction as the inverse of addition.

a) $\frac{3}{8} - \frac{4}{5} = \frac{3}{8} + \frac{-4}{5} = \frac{5(3) + 8(-4)}{40} = \frac{15 + (-32)}{40} = \frac{-17}{40}$ or $-\frac{17}{40}$

b) $-\frac{3}{5} - \frac{4}{7} = \frac{-3}{5} + \frac{-4}{7} = \frac{7(-3) + 5(-4)}{35} = \frac{-21 + (-20)}{35} = \frac{-41}{35}$ or $-\frac{41}{35}$

c) $-\frac{1}{3} - \frac{-4}{5} = \frac{-1}{3} + \frac{+4}{5} = \frac{5(1) + 3(4)}{15} = \frac{-5 + 12}{15} = \frac{7}{15}$

Multiplication of Rationals

Concept: When two rationals are multiplied such as $\frac{a}{b}$ and $\frac{c}{d}$, the product is $\frac{ac}{bd}$.

a) $(\frac{2}{3})(\frac{5}{7}) = \frac{(2)(5)}{(3)(7)} = \frac{10}{21}$

b) $(\frac{3}{4})(-\frac{4}{5}) = \frac{(3)(-4)}{(4)(5)} = \frac{-12}{20}$ or $-\frac{12}{20} = -\frac{3}{5}$

c) $(-\frac{1}{3})(-\frac{2}{5}) = \frac{(-1)(-2)}{(3)(5)} = \frac{+3}{15}$ or $\frac{3}{15} = \frac{1}{5}$

Students should be able to formulate the following rules:

1. If the rationals being multiplied are all positive, the product will be positive.

2. If there are an even number of negative rationals, among the rationals being multiplied, the product will be positive.

3. If there are an odd number of negative rationals, among the rationals being multiplied, the product will be negative.

Concept: The product of a number and its multiplicative inverse (or reciprocal) is always 1.

The multiplicative inverse of a negative rational then must be negative. For example, if the negative rational is

$$-\frac{2}{3} \text{ , its multiplicative inverse is } -\frac{3}{2}$$

to check: $(-\frac{2}{3})\cdot(-\frac{3}{2}) \overset{?}{=} 1$

$$\frac{(-2)(-3)}{(3)(2)} = \frac{6}{6} = 1$$

Any rational equivalent to zero does *not* have a multiplicative inverse since division by zero is not permitted.

Division of Rationals

Concept: In the set of rationals, division is defined as the inverse operation of multiplication.

a) $\frac{2}{3} : \frac{5}{6} = \square$ or $\frac{\frac{2}{3}}{\frac{5}{6}} \cdot 1 = \frac{\frac{2}{3}}{\frac{5}{6}} \cdot \frac{\frac{6}{5}}{\frac{6}{5}} = \frac{(\frac{2}{3})\cdot(\frac{6}{5})}{1} = \frac{12}{15} = \frac{4}{5}$

$\square\cdot(\frac{5}{6}) = \frac{2}{3}$

$\boxed{(\frac{2}{3}) \cdot (\frac{6}{5})} \cdot (\frac{5}{6}) = \frac{2}{3}$

$\square = \frac{12}{15} = \frac{4}{5}$

b) $(\frac{-4}{5}) : (\frac{3}{4}) = \square$ or $\frac{-\frac{4}{5}}{\frac{3}{4}} \cdot 1 = \frac{\frac{-4}{5}}{\frac{3}{4}} \cdot \frac{\frac{4}{3}}{\frac{4}{3}} = \frac{(\frac{-4}{5})\cdot(\frac{4}{3})}{1} = \frac{-16}{15}$

$\square\cdot(\frac{3}{4}) = \frac{-4}{5}$

$\boxed{(\frac{-4}{5}) \cdot (\frac{4}{3})} \cdot (\frac{3}{4}) = -\frac{4}{5}$

$\square = \frac{-16}{15}$ or $-\frac{16}{15}$

c) $(\frac{-5}{11}) \div (\frac{-2}{3}) = \square$ or $\dfrac{\frac{-5}{11}}{\frac{-2}{3}} \cdot 1 = \dfrac{\frac{-5}{11}}{\frac{-2}{3}} \cdot \dfrac{\frac{-3}{2}}{\frac{-3}{2}} = \dfrac{(\frac{-5}{11}) \cdot (\frac{-3}{2})}{1} = \dfrac{15}{22}$

$\square \cdot (\frac{-2}{3}) = -\frac{5}{11}$

$\boxed{(\frac{-5}{11}) \cdot (\frac{-3}{2})} \cdot (\frac{-2}{3}) = -\frac{5}{11}$

$\square = \frac{15}{22}$

Concept: Students should recognize that the rules for signs in division are the same as those for multiplication since the use of the multiplicative inverse changes the form from division to a corresponding multiplication.

Summary of the Properties of all the Rationals. The following is a summary of the properties of the rationals:

(a) Closure for addition, subtraction, multiplication, and division
(b) Commutative property under addition and multiplication
(c) Associative property under addition and multiplication
(d) Multiplication distributive over addition and subtraction
(e) Identity element for addition and for multiplication
(f) Inverse property for addition and multiplication

Negative Exponents

Negative exponents may now be introduced. Review the rules for multiplication and division of powers of the same base. Extend the use of the rules to include examples where the greater power is in the denominator

$$\begin{tikzcd} \frac{1000}{10,000} = \frac{1}{10} \arrow[d, leftrightarrow] \\ \frac{10 \times 10 \times 10}{10 \times 10 \times 10 \times 10} = \frac{10^3}{10^4} = 10^{3-4} = 10^{-1} \end{tikzcd}$$

$$\begin{tikzcd} \frac{10}{1000} = \frac{1}{100} \arrow[d, leftrightarrow] \\ \frac{10}{10 \times 10 \times 10} = \frac{10^1}{10^3} = 10^{1-3} = 10^{-2} \end{tikzcd}$$

Exponential forms of places to the right of the decimal:

$$.\ \underline{10^{-1}}\ \underline{10^{-2}}\ \underline{10^{-3}}\ \ldots$$

At this point students should see that the ones place is the balancing point of our number system

...	10^5	10^4	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	...
	Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths	Ten Thousandths	

Concept: In base ten numerals, including decimal fractions, any place is one tenth of the place to its left and ten times the place to its right.

Scientific Notation

Scientific notation can now be extended to decimal fractions.

$.5 = 5 \times 10^{-1}$

$.01 = 1 \times 10^{-2}$

$.00236 = 2.36 \times 10^{-3}$

Write the following numbers in scientific notation.

a) 5,500,000 b) 639 c) .4372 d) .0057

Answers:

a) 5.5×10^6 b) 6.39×10^2 c) 4.372×10^{-1} d) 5.7×10^{-3}

Measurement

Concept: All measurements are only approximations of the real measure.

A measurement can not be exact. A measurement depends on a calibrated instrument of some kind and a person reading the instrument, so there are chances for making mistakes, one in the calibration and one in the inaccurate reading of the measuring device. But more fundamentally for each calibration there always is a range of measures which are assigned to that calibration.

Concept: Precision is dependent upon the size of the unit of measure.

The precision increases as the size of the unit of measure decreases; in other words, the smaller the unit of measure, the greater the precision.

i.e. A ruler which is calibrated in $\frac{1}{16}$ inch units is more precise than a ruler which is calibrated in $\frac{1}{4}$ inch units.

Error of Measure is the difference between the actual measure and the measure as it is read. This error may not necessarily be a mistake, but may be due to faulty calibration and/or the approximation which must be made in reading. Since all measurement is approximate, we must expect some error.

Concept: Absolute error is ± (read: plus or minus) one half the smallest unit calibrated.

i.e. A ruler calibrated in one tenth inch units has an absolute error of ± .05.

Absolute error may also be referred to as *possible error*, *greatest possible error*, or *tolerance*. It cannot be determined unless the unit of measure used is known.

Concept: The relative error is the ratio of the absolute error to the total measure.

Thus, the relative error, using the same measuring device, decreases as the total measure increases.

i.e. Using a ruler calibrated in $\frac{1}{4}$ units a line measures $2\frac{1}{2}$ in. The relative error then is $\frac{1}{8} : 2\frac{1}{2}$ or 1:20.

Using the same ruler, a line measures 10 in. the relative error then is $\frac{1}{8} : 10$ or 1:80.

Concept: Accuracy is related to relative error; as the relative error decreases, the accuracy increases.

i.e. Using a ruler calibrated to $\frac{1}{2}$ in. units a 10 in. line is measured.

1/4 : 10 or R.E. = 1:40

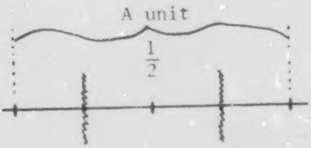
Using a ruler calibrated to $\frac{1}{2}$ in. units a 5 in. line is measured.

1/4 : 5 or R.E. = 1:20

The measurement of the 10 in. line is more *accurate* since the relative error is less.

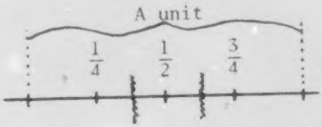
Concept: Measures which are less than half the distance between two calibration marks away from a particular calibration mark, on either side of it, are all assigned the value of that particular calibration mark.

(A)



A measure of $\frac{1}{2}$ unit on a unit marked to $\frac{1}{2}$ units would be assigned all points of the segment between the $\frac{1}{4}$ and $\frac{3}{4}$ units, as shown by the wavy lines.

(B)



A measure of $\frac{1}{2}$ unit on a unit marked to $\frac{1}{4}$ units would be assigned to all points of the segment between $\frac{3}{8}$ and $\frac{5}{8}$ as shown by the wavy lines.

Students should be shown that there is a difference between a reading of a $\frac{1}{2}$ unit measured with rulers calibrated with different degrees of precision.

Ruler A $\frac{1}{4} < \frac{1}{2} < \frac{3}{4}$

Ruler B $\frac{3}{8} < \frac{1}{2} < \frac{5}{8}$

The method for showing the difference might be to keep as a denominator the unit of calibration.

i.e. For Ruler A, $\frac{1}{2}$ would be used; but for Ruler B, $\frac{2}{4}$ would be used to show that the calibration was $\frac{1}{4}$ inches.

1. *What is the absolute error for the measuring devices calibrated as shown in the following exercises?*

   a) Ruler to $\frac{1}{8}$ inches b) Cup to $\frac{1}{3}$ cup c) Ruler to $\frac{1}{10}$ inches.
   d) Altimeter to 10 feet

Answers:

   a) $\pm \frac{1}{16}$ in. b) $\pm \frac{1}{6}$ cup c) $\pm \frac{5}{100}$ in. or $\pm \frac{1}{20}$ in. or $\pm$ .05 inches d) $\pm$ 5 feet

2. *Find the relative errors of each of the following:*

   a) 36 ft. b) 360 yd. c) 2.5 mm. d) .126 mm.

Answers:

   a) $\frac{1}{2} : 36 = 1{:}72$ b) $5{:}360 = 1{:}72$ c) $\frac{.05}{2.50} = \frac{5}{250} = 1{:}50$
   d) $\frac{.0005}{.126} = \frac{5}{1260} = 1{:}252$

3. *Which measurement in each pair has (1) the greater precision; (2) the greater accuracy?*

   a) 8.2 yds., $3\frac{1}{3}$ yds. b) .75 inches, 23.0 feet
   c) .32 cm., .46 cm.

Answers:

	Precision	Accuracy
a)	first	first
b)	first	second
c)	same	second

Significant Digits

Significant digits are important to denote the accuracy of a measurement. Regardless of the placement of the decimal point, a greater number of significant digits show a more accurate measurement. The error is taken as not greater than one-half the place value of the last significant digit.

Concept: All non-zero digits are significant. Zeros between-non-zero digits are significant. Final zeros of a measurement expressed as a decimal fraction (or mixed number) are significant.

Zeros directly following the decimal point in a number less than one are *not* significant. Final zeros in an integer are not significant unless there is an indication of significance such as plus or minus an error. The numerals we use need to be supplemented with information about the measurement unit in order to have an accurate meaning.

Exercises (Units of measurement will be omitted in these exercises)

1) *How many significant digits are there in each of the following measurements?*

(a) 482 (b) 30619 (c) 3.14 (d) 2.005 (e) 200
(f) 200.0 (g) .0016 (h) .000150

Answers:

(a) 3 (b) 5 (c) 3 (d) 4 (e) 1 (f) 4 (g) 2 (h) 3

The number of significant digits can be shown in scientific notation by using an appropriate number of decimal places:

300, which has one significant digit, 3×10^2

340, which has two significant digits, 3.4×10^2

4056, which has four significant digits, 4.056×10^3

5000, which has one significant digit, 5×10^3

Square Root

Concept: The square root of a number is one of two equal factors of the number.

The radical sign $\sqrt{\ }$ means the positive square root and is also referred to as the principal root.

The square of a number n is $n \times n = n^2$

Perfect squares are squares of the integers

$$1^2 = 1 \qquad (-1)^2 = (-1)\cdot(-1) = 1$$

$$2^2 = 4 \qquad (-2)^2 = (-2)\cdot(-2) = 4$$

$$3^2 = 9 \qquad (-3)^2 = (-3)\cdot(-3) = 9$$

etc. etc.

The squares of the integers are always positive numbers.

Concept: A positive number then has two square roots, one which is positive and one which is negative.

The square root of 25 = +5 or -5, commonly written ± 5, which is read plus or minus five.

However $\sqrt{25} = +5$, the principal root only.

There are square root tables in many textbooks which should be utilized in the initial work in this topic, starting with the perfect squares, then progressing to whole numbers.

There are several processes which may be taught for the extraction of square root: among them the standard algorithm, the "guess and multiply" or "trial and error" method, and the averaging method.

Anyone who knows the meaning of square root and can handle long division may quickly be taught the averaging method. For example, we wish to find the square root of 482. We assume that the square root is 20. If this assumption is correct, the quotient of 482 ÷ 20 should be 20. If 20 is too small, the quotient will be larger than the actual square root; while if 20 is too large, the quotient will be smaller than the actual square root. When the divisor, 20, and the quotient are averaged, the average should approximate more closely the square root.

Example shows how the square root of 482 may be obtained by the averaging method.

```
    24.1
20)482.0
   40
    82
    80
     20
```

The average of 20 and 24 is 22, which is used as the next divisor.

```
    21.90
22)482.00
   44
    42
    22
    200
    198
     20
```

The average of 22 and 21.90 is 21.95 which is correct to the nearest hundredth. This can be checked by dividing 482 by 21.95

```
       21.95
2195.)48200.00
       4390
        4300
        2195
        21050
        19755
        12950
        10975
         1975
```

If a more accurate square root is needed, the division by 21.95 may be continued for several more places, and the divisor and quotient again averaged.

In the preceding example the guess was too small. Since we already know the square root to the nearest hundredth, let us now make a guess which we know is too large to show what happens. Let us assume 25 is the square root.

```
   19.2            19
25)482.0           25
   25             |44
   232             22
   225
     70
     50
```

```
   21.9            21.9
22)482.0           22
   44             |43.9
    42             21.95
    22
    200
    198
```

In order to keep the division from being carried out further than is useful we may point out the following: In general, if the divisor and quotient agree in the first n digits 2n digits should be kept in the quotient. If the division was carried further the rest of the digits should be changed to zero. Thus, 20 and 24.1, have only the first digit in common, n=1. Therefore, 2n or 2 digits will be kept in quotient. Thus 24.1 becomes 24 and then is used for averaging. In the first division it may be necessary to keep an extra digit or two if the first guess is too far from the square root, leading to no agreement in the digits.

VIII. THE SET OF REAL NUMBERS

Concept: Any number that can be expressed as the ratio or quotient of two integers a and b, that is, in the form $\frac{a}{b}$ ($b \neq 0$) is a rational number.

A rational may be expressed in decimal notation. The rational $\frac{a}{b}$ is the quotient resulting from the division a:b. The rational $\frac{a}{b}$ may be expressed in decimal notation by performing the division a:b. For example, $\frac{5}{8} = 5:8 = 0.625$

It is possible that the decimal does not terminate. Examples of rationals which form non-terminating decimals are as follows:

(a) $\frac{1}{3} = 0.33333\ \ldots$

(b) $\frac{3}{11} = 0.2727272727\ \ldots$

(c) $\frac{4}{9} = 0.44444\ \ldots$

(d) $\frac{2}{7} = 0.2857142857142\ \ldots$

Concept: If a rational does not form a terminating decimal, it will form a non-terminating decimal in which a digit or group of digits keeps repeating.

When performing the division a:b, the decimal must either terminate or begin repeating by the "b"th digit after the decimal point. It may begin repeating sooner, but it must begin repeating by the "b"th digit after the decimal point. For example, in performing the division 2:7, each time a digit in the answer is determined, the remainder may be 0, 1, 2, 3, 4, 5, or 6. If the remainder is zero, the decimal terminates at that point. The remainder may be different for the first six digits but when the seventh digit has been determined, the remainder must be one of the previous remainders. When this occurs, the decimal must begin to repeat. Therefore, every rational forms either a terminating decimal or a non-terminating decimal that repeats.

Concept: Any decimal that does not terminate and does not repeat is an expression for a number which is not a rational number.

Three examples of non-terminating, non-repeating decimals are:

(a) 0.123456789101112 ...
(b) 1.122334455
(c) 0.112123123412345 ...

Concept: A number that can be represented on a number line but is not a rational number is called an irrational number.

Numbers such as $\sqrt{2}$ and π are not rational numbers. They cannot be expressed as the quotient or ratio of two integers and they form non-terminating and non-repeating decimals.

An important mathematical question that once faced mathematicians was whether or not irrational numbers were represented by points on a number line, or would the set of rationals account for all points on a number line. There is an infinite number of rationals, and between any two rationals there is an infinite number of rationals. If the set of rationals is represented by points on a number line, it might seem that every point on the number line would represent a rational number. The ancient Greeks discovered that there were points on a number line that represented numbers which were not rational numbers. This can be demonstrated in an application of the Pythagorean theorem.

The Pythagorean theorem states that the sum of the squares of the lengths of the legs of any right triangle is equal to the square of the length of the hypotenuse: $a^2 + b^2 = c^2$, where a and b are the lengths of the legs and c is the length of the hypotenuse. If each leg is one unit in length, then

$$1^2 + 1^2 = c^2$$

$$1 + 1 = c^2$$

$$2 = c^2$$

$$\sqrt{2} = c$$

The hypotenuse is $\sqrt{2}$ units in length. A line segment equal to the length of the hypotenuse can be marked off on a number line. If one end of the segment is at the zero point on the number line, the other end of the segment will coincide with a point on the number line. This point must represent the number $\sqrt{2}$. When it was proven that $\sqrt{2}$ was an irrational number, this meant that there was a point on the number line which represented an irrational number.

Other irrational numbers such as π are represented by points on a number line. π is defined as the ratio of the circumference of any circle to its diameter. For any circle, $\frac{C}{D} = \pi$. The circumference is equal to πD. If the diameter is one unit in length, the circumference is π units in length. The circumference of a circle whose diameter is one unit may be marked off on a number line, therefore, π is represented by a point on a number line.

Concept: The set of numbers consisting of all rationals and irrationals is called the set of real numbers.

Every real number is represented by a point on a number line and every point on a number line represents a real number. There is a one-to-one correspondence between the points on a number line and the set of real numbers. For this reason, the number line is called the real number line.

IX. RATIO, PROPORTION, PER CENT, VARIATION

Ratio

Concept: A ratio is a comparison of two numbers by division.

The ratio of the number a to the number b $(b \neq 0)$ is the quotient $\frac{a}{b}$. This ratio may also be written as a:b. The ratio 6 to 11 may be written as $\frac{6}{11}$ or 6:11.

The ratio is read: 6 to 11, regardless of the form in which it is written.

In comparing numbers resulting from measurements, care must be taken that both numbers in the ratio are measures in the same units. The ratio then is merely a number.

Problem: Find the ratio of 1 inch to 2 feet.

Answer: $\frac{1 \text{ in.}}{2 \text{ ft.}} = \frac{1 \text{ in.}}{24 \text{ in.}}$ The ratio is 1 to 24

Ratios may be expressed in reduced form. The ratio a:b may be expressed in the rational form $\frac{a}{b}$. The concept of reducing a rational to lowest terms applies to ratios. The ratio $\frac{a}{b}$ is said to be reduced to lowest terms if a and b have no common factors other than the number 1. A ratio is reduced by canceling all factors that are common to both the numerator and denominator, applying the principle of equality of rationals $\frac{ac}{bc} = \frac{a}{b}$.

Problem: Reduce the ratio 5:15

Answer: $5:15 = \frac{5}{15} = \frac{1}{3} = 1:3$

Further work with ratios in the eighth grade may include the comparison of ratios. The numerator and denominator of any given ratio may be multiplied by the same factor in forming an expression equivalent to the given expression (again zero is excluded). Given two ratios, the method used to determine which is the greater ratio is the same method used in determining which of two given rational numbers is greater. The ratios are first each written in the form of a positive rational with a positive denominator. Expressions equivalent to the given expressions are determined such that the changed expressions both have the same denominator. The expression with the greater numerator is the expression for the greater ratio.

Problem: Determine which of the two given ratios, $\frac{14}{16}$ or $\frac{21}{27}$, is the greater ratio.

Answer: The following procedure is carried out: $\frac{14}{16} = \frac{63}{72}$; $\frac{21}{27} = \frac{56}{72}$; and $\frac{63}{72} > \frac{56}{72}$. Therefore, $\frac{14}{16}$ is a ratio which is greater than the ratio $\frac{21}{27}$.

Concept: A rate is a ratio which compares two numbers which result from measurements having different units.

Some ratios are the comparison of two numbers which are measures in different units. Such ratios are usually called *rates* and include a specific phrase of the form "*a per b*." The rate of speed of an automobile is the ratio of the distance traveled to the time taken in traveling that distance. If an automobile travels 200 miles in 4 hours, the rate of speed of the automobile is $\frac{200}{4}$ or $\frac{50}{1}$ mph. Many rates are expressed as ratios whose denominator is 1, and the units are indicated. The ratio $\frac{50}{1}$ mph above is read "50 miles per hour."

Proportion

Two ratios $\frac{a}{b}$ and $\frac{c}{d}$ (or a:b and c:d) are equal if and only if ad = bc. The fact that two ratios are equal may be expressed as $\frac{a}{b} = \frac{c}{d}$ or as a:b = c:d.

Concept: An expression for the fact that two ratios are equal is called a proportion.

A proportion may be written $\frac{a}{b} = \frac{c}{d}$ or a:b = c:d and in either example it is read "a is to be as c is to d." Note: Division by zero is excluded.

Thus, $\frac{4}{7} = \frac{12}{21}$ or 4:7 = 12:21; and $\frac{9}{5} = \frac{18}{10}$ or 9:5 = 18:10 are proportions.

Concept: If three of the four terms in a proportion are given, the fourth term can be determined.

Proportions with one unknown term can be solved by the open sentence approach. Applying the principle that if a:b = c:d (or $\frac{a}{b} = \frac{c}{d}$) then ad = bc, a simpler equation can be written for any given proportion, and this second equation can be solved for the unknown term. For example, if 3:5 = 21:x (or $\frac{3}{5} = \frac{21}{x}$), then 3x = 105. This equation can then be solved by dividing each side of the equation by 3, resulting in the solution x = 35. The answer can be determined to be a correct solution by substituting it for the unknown term in the original proportion.

$$\begin{aligned} 3:5 &= 21:35 \\ (3)(35) &= (5)(21) \\ 105 &= 105 \text{ true} \end{aligned}$$

This is called checking the solution.

Problem: If an ice cream mix uses 9 eggs to produce 3 quarts, how many eggs will be used to produce 1 gal?

Answer: 1 gal. = 4 qts.

$$\frac{9}{3} = \frac{x}{4}$$

$$3x = 36$$

$$x = 12$$

Concept: In a scale drawing, the ratio of a dimension on the drawing to the actual dimension represented by the drawing is equal to the ratio of any other dimension on the drawing to the corresponding dimension of the object represented by the drawing.

For example, $\frac{\text{length of drawing}}{\text{length of actual room}} = \frac{\text{width of drawing}}{\text{width of actual room}}$. Rates as well as ratios may be used to set up the proportion, that is, the numerator and the denominator of each fraction may be measured in different units provided that both numerators are measured in the same unit and both denominators are measured in the same unit. Also, the ratio of any two dimensions on the drawing equals the ratio of the corresponding dimensions of the actual room in the same order. For example,

$$\frac{\text{length of drawing}}{\text{width of drawing}} = \frac{\text{length of room}}{\text{width of room}}$$

Per Cent

Concept: A per cent is a rate in which the denominator is always 100.

The term per cent is written as two words; "percentage" is written as one word and means the part of the whole which is represented by the per cent. In baseball standings, the "pct." or percentage column is really the per cent of the games won by the team, expressed as a decimal fraction rounded to 3 places. *How are per cents used?* Some common uses of per cents would be:

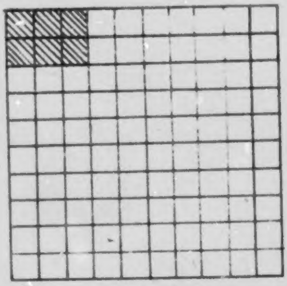
(a) discount - 25% off
(b) an interest rate per annum (year) = 4% interest
(c) a commission for selling goods or services - 5% commission
(d) the strength of a solution - 3% hydrogen peroxide solution
(e) a relative change - 15% increase in salary
(f) baseball team percentage
(g) batting averages - a player batting 0.300 = often read as "300"
(h) a means of showing statistics - 70% of the graduation class went to an institution of higher learning or soap is $99 \frac{44}{100}$ % pure (pure what?)

What does "per cent" mean? This should lead to a discussion of the literal meaning "divide by 100" or hundredths, drawings of what it means and different ways of writing a per cent. We would want the pupils to come up with a list of numerals to show; for example, six per cent:

$6\%:0.06, \frac{6}{100}, 6:100, 100\overline{)6},$

$6:100$, 6 out of 100, $(6 \times \frac{1}{100})$,

(6×0.01). A picture is shown at the right.



Why is per cent used instead of just writing a decimal or common fraction? We use percents because:

(1) per cent gives us a common denominator so we can compare different fractions
(2) many people do not care to work with decimals or common fractions when they can work with whole numbers with the per cent symbol (%) annexed, thus, 0.28 or $\frac{7}{25} = 28\%$.

Concept: To express the ratio $\frac{a}{b}$ *in per cent notation the ratio may first be written as an equivalent ratio whose denominator is 100, such as* $\frac{c}{100}$. *This ratio may then be written in the per cent notation as c%.*

For example, the ratio of 4 to 25 is $\frac{4}{25}$. An equivalent expression for $\frac{4}{25}$ with a denominator of 100 is $\frac{16}{100}$. This may be written in per cent notation as 16%. The fraction $\frac{16}{100}$ written in decimal notation is 0.16. By observing the relation between a ratio whose denominator is 100, its decimal equivalent, and the corresponding per cent notation, the pupil can easily be shown the method of changing a decimal to a per cent. For example,

$$\frac{7}{100} = 0.07 = 7\%$$

$$\frac{19}{100} = 0.19 = 19\%$$

$$\frac{131}{100} = 1.31 = 131\%$$

From such examples the pupil can see that to change a decimal to per cent notation, the decimal point in the per cent notation is two places further to the right than in the decimal notation.

The pupil should already be familiar with the procedure for changing a rational to a decimal by dividing the numerator by the denominator. If the resulting decimal does not terminate within two decimal places it is common practice to carry out the division to two decimal places and then express any remainder in fractional form. For example, $\frac{1}{3} = 0.33\frac{1}{3}$; $\frac{5}{8} = 0.62\frac{1}{2}$; $\frac{1}{200} = 0.00\frac{1}{2}$.

The resulting decimal can then be written in per cent notation.

$$0.33\frac{1}{3} = 33\frac{1}{3}\%$$

$$0.62\frac{1}{2} = 62\frac{1}{2}\%$$

$$0.00\frac{1}{2} = \frac{1}{2}\%$$

Therefore, any ratio can be changed to per cent notation by changing it to equivalent decimal notation and this to per cent notation. Except in the introductory work of this unit, it is not necessary to change the ratio to an equivalent ratio with denominator 100.

Concept: Given two numbers, a and b, it is possible to find what per cent a is of b by expressing the ratio $\frac{a}{b}$ as an equivalent decimal and then changing this decimal to per cent notation.

For example, when a store sells merchandise, its per cent profit is usually defined as the ratio of the net profit to the total cost, this ratio being expressed as a per cent. If a profit of \$216 is realized on goods that cost \$864, then the per cent profit is $\frac{\$216}{\$864} = 0.25 = 25\%$.

Per cent profit may be defined in other ways such as the ratio of gross profit to cost of goods sold. In presenting exercises of this type to the pupils, care must be taken to make it very clear which two numbers are to be used in forming the ratio.

A second aspect of the concept of per cent pertains to calculating a per cent of a number. If a pupil were asked to find $\frac{3}{4}$ of 416, he would probably realize that this was a multiplication problem and he would proceed to calculate the product of $\frac{3}{4}$ and 416. If he is asked to find 0.75 of 416, this also is a multiplication problem and is solved by calculating the product (.75)(416). If either the $\frac{3}{4}$ or the 0.75 is replaced by the equivalent notation, 75%, the problem involves multiplication, that is, finding the product (75%)(416). Finding 75% of 416 is a multiplication problem. In multiplying by a per cent, the per cent is first changed to the equivalent decimal or fraction and then the multiplication performed.

Problem: Find 16% of 85.

Answer: The 16% may be changed to the equivalent decimal 0.16 and the multiplication (0.16)(85) performed, giving the product 13.6.

When a decimal is written in equivalent per cent notation, the decimal point is two places further to the right than in the given decimal. Therefore, in changing a per cent to equivalent decimal notation, the decimal point will be two places further to the left than in the given per cent. If desired, the decimal may be then changed to a ratio or fraction. Very often it is of benefit to memorize a list of commonly used fractions and their equivalent per cent expressions.

A third aspect of the concept of per cent pertains to determining a number when a per cent of it is given. One method of solving such problems is by the use of simple open sentences. For example, consider the problem of determining a number when it is known that 18% of the number is 90. This information may be summarized by the equation $0.18x = 90$, or the equation $\frac{18x}{100} = 90$. In the first equation, the 18% has been written in its equivalent decimal form and in the second equation it is written in equivalent fractional form. Solving either of these equations gives the solution 500.

The equation $0.18x = 90$ is solved by dividing each side by 0.18. The equation $\frac{18x}{100} = 90$ may be solved by multiplying each side by 100 and then dividing each side by 18, or by first dividing each side by 18 and then multiplying each side by 100.

There are advantages in solving all types of per cent problems by the use of a proportion. In this approach the % is first changed to a common fraction form.

Problem: Find 25% of 84.

Answer: $25\% = \frac{25}{100} = \frac{1}{4}$

$$\frac{25}{100} = \frac{n}{84} \quad \text{or} \quad \frac{1}{4} = \frac{n}{84} \qquad \text{Ck.} \quad \frac{1}{4} \overset{?}{=} \frac{21}{84}$$

$$100n = 25(84) \qquad 4n = 84 \qquad 84 = 84$$

$$100n = 2100 \qquad n = \frac{84}{4} = 21$$

$$n = \frac{2100}{100} = 21$$

Problem: Find the number if 15% of it is sixty.

Answer:

$$15\% = \frac{15}{100} = \frac{3}{20} \qquad \text{Ck.} \quad \frac{3}{20} \overset{?}{=} \frac{60}{400}$$

$$\frac{3}{20} = \frac{60}{n}$$

$$1200 = 1200$$

$$3n = 1200, \; n = 400$$

Problem: What % of 45 is 9?

Answer:

$$\frac{n}{100} = \frac{9}{45}$$

$$45n = 900$$

$$n = \frac{900}{45} = 20, \; \frac{20}{100} = 20\%$$

Verbal problems involving per cent may be solved as general proportions.

Discount: A coat is on sale at $33\frac{1}{3}\%$ off the regular price of \$75. (a) What is the amount of discount? (b) What is the sale price of the coat?

Answer:

a) $33\frac{1}{3}\% = \frac{1}{3}$

$$\frac{1}{3} = \frac{n}{75}$$

$$3n = 75$$

$$n = \$25$$

b) Sale price = \$75-\$25 = \$50

If only question (b) is asked, students should be led to realize that $33\frac{1}{3}\%$ off the original price means also $66\frac{2}{3}\%$ of the original price is left as the cost of the coat then:

$$66\tfrac{2}{3}\% = \frac{2}{3}$$

$$\frac{2}{3} = \frac{x}{75}$$

$$3x = 150$$

$x = \$50$ the sale price of the coat.

Per cent change: (The change is dependent upon what a number was before the change.) A club had 50 members at the beginning of the school year, but by the middle of year only 35 members were still active. Find the % decrease in the membership.

Answer:

$$50-35 = 15$$

$$\frac{n}{100} = \frac{15}{50}$$

$$50n = 15(100)$$

$$n = \frac{15(\overset{2}{\cancel{100}})}{\cancel{50}}$$

$n = 30$ ∴ 30% decrease in membership.

General: There were 100 reserved seat tickets and 300 general admission tickets sold to a play. a) Find the % of reserved seat tickets sold. b) Find the % of general admission tickets sold.

Answer: Total sold is 100 + 300 = 400 tickets

a) $\frac{n}{100} = \frac{100}{400}$

$400n = 100(100)$

$n = \frac{100(100)}{400}$

$n = 25 \quad \therefore \frac{25}{100} = 25\%$

b) $\frac{n}{100} = \frac{300}{400}$

$400n = 100(300)$

$n = \frac{100(300)}{400}$

$n = \frac{300}{4} = 75$

$\frac{75}{100} = 75\%$ or $100\% - 25\% = 75\%$

Commission: A real estate salesman works on an 8% commission. How much is the commission on the sale of a $20,000 house?

Answer: $8\% = \frac{8}{100} = \frac{2}{25}$

$\frac{8}{100} = \frac{n}{20,000}$ or $\frac{2}{25} = \frac{n}{20,000}$

$100n = 8(20,000)$ $\qquad 25n = 2(20,000)$

$n = \frac{8(20,000)}{100}$ $\qquad n = \frac{40,000}{25} = \$1600$

$n = \$1600$ amount of Commission

A key concept in interest problems is the utilization of the time factor. There should be an ample discussion of the rate of interest related to a year. Then develop the idea that 6 mo. interest is only $\frac{1}{2}$ that for a year, 90 days interest is only $\frac{1}{4}$ that for a year, 2 years interest is twice that for a year, etc.

Interest: A man borrowed \$3000 for 4 years at an interest rate of 7%. Find the interest for 1 year, and then for four years.

Answer: $\frac{7}{100} = \frac{x}{3000}$

$$100x = 7(3000)$$

$$x = \frac{7(3000)}{100}$$

x = \$210 int. for 1 year

Int. for 4 years

4 x \$210 = \$840

Variation

If an automobile is driven along a highway at an average speed of 50 miles per hour, the distance traveled increases as the time traveled increases. As the number of hours spent in traveling changes, the distance traveled changes. The time and distance traveled are called variables and are interdependent. As the automobile is driven along the highway at an average speed of 50 miles per hour, it will have traveled 50 miles at the end of the first hour, 100 miles at the end of the second hour, and 150 miles at the end of the third hour. The distance traveled is always the product of the rate and the number of hours traveled. If d represents the distance traveled and t represents the time traveled, then $d = 50t$. Dividing both sides of the equation by t, the equation $\frac{d}{t} = 50$ results. The ratio of the distance and the time is a constant.

Concept: If two variables vary in such a way that their ratio is a constant, each of the variables is said to vary directly at the other variable. This is an example of direct variation.

Some examples of direct variation are:

(a) The total cost of a number of identical items varies directly as the number of items purchased if the cost of each item is a constant.

(b) The total interest on a loan varies directly as the amount of money borrowed if the interest rate is constant.

(c) The total wages a worker earns varies directly as the number of hours he works providing his hourly wage rate remains constant.

(d) The area of a rectangle varies directly as the length of the rectangle if the width remains constant.

(e) The circumference of a circle varies directly as the diameter of the circle.

Problem: If t represents the number of hours spent in traveling at 60 miles per hour and d represents the distance traveled during time t, make a 3-rowed table for $d = 60t$, listing t, d, and the ratio $\frac{d}{t}$ for $t = 4, 5, 6, 8, 10$.

Answer:

t	4	5	6	8	10
d	240	300	360	480	600
$\frac{d}{t}$	60	60	60	60	60

The ratio $\frac{d}{t}$ is constant, therefore this is direct variation.

Denominate numbers: Review equivalent measures and changing from larger units to smaller units i.e. feet to inches; and from smaller units to larger i.e. minutes to hours. Remainders may be expressed either in the smaller unit or in decimal fraction or common fraction part of the larger unit.

The concept of direct variation can be applied to equivalent measures using pairs of denominations.

Problem: If f represents the number of feet and i represents the number of inches in the same length, fill in the table and show that this is an example of direct variation.

f	3		$4\frac{1}{2}$		5.5
i		108		15	
$\frac{i}{f}$					

Answer:

f	3	9	$4\frac{1}{2}$	$1\frac{1}{4}$	5.5
i	36	108	54	15	66
$\frac{i}{f}$	12	12	12	12	12

Since the ratio $\frac{i}{f}$ is constant (it is 12) then this is an example of direct variation.

X. GEOMETRY

In mastering the material in this unit, the pupil is embarking on a new adventure. While much of the geometry studied in this unit has valuable applications in daily living, some is pure mathematics. Instead of dealing with numbers of physical objects, the pupils now deal with mental concepts such as point, line, and plane.

If properly taught, this unit on geometry can be a fascinating and stimulating adventure in mathematics. If improperly taught, it can become a humdrum process of memorizing 150 terms and formulas.

The pupil must be allowed the opportunity of experiencing the thrill of mathematical discovery. Give him only necessary definitions and then lead him by skillful questioning and presentation of problems to discover for himself geometric concepts and relationships.

The normal seventh or eighth grader does not have the tools of logical proof, but even at this level the pupil can be taught at least one method of disproof - disproof by counterexample.

Concept: A statement can be disproved by producing a single counterexample.

The statement: "Any two triangles are congruent if three angles of one are equal to three angles of the other" can be easily disproved. Construct two triangles whose corresponding angles are equal but let one triangle have longer sides than the other. The two triangles cannot be made to coincide and so are not congruent. The statement is therefore false. Disproof by counterexample is a simple but effective way of stimulating mathematical thinking and offering a challenge to even the brightest students.

This unit presents the basic problem of being mathematically correct. However, teaching above the ability level of the pupils to be precisely mathematically correct and exact in all topics is not advisable. It must be remembered that *not* all students are going to be mathematicians.

Each student should have for his individual use, a pair of compasses; a protractor and a straight edge. A variety of types of protractors should be available for student use. Ideally the straight edge should have no markings, but in classroom use, a ruler suffices.

Undefined Terms

Dictionary definitions are usually circular. The dictionary definition of "equal" is "same." The definition of "same" is "identical." The definition of "identical" is "same or equal." If the meaning of one of these words was not already known, the dictionary would not be of much help in explaining the meanings of the other words.

In any series of definitions, the meaning of certain words must already be known. It is necessary to develop an intuitive knowledge of the meaning of these words by observing how they are used in the language. If the meaning of certain basic words can be accepted intuitively, then the other terms can be defined in terms of these basic words.

The study of geometry begins with the selection of basic words, the meaning of which will be arrived at intuitively. The undefined terms are *point*, *straight line*, and *plane*. The meaning of these words is made apparent by observing how the words are used and by examining their basic properties and characteristics.

Point

Concept: A point has no dimensions. It has neither length, nor width, nor height. It has a single property and that is the property of position.

A point can be described by its position. A point can be described by its position on a one-dimensional line, a two-dimensional plane, a three-dimensional space, or in a space of more than three dimensions which mathematicians have created.

A point is a mental concept. Its position can be represented by a dot on a piece of paper or by a dot on a chalkboard. The dot is not the point. It represents the position of the point. The dot is nothing but a little ink or a small bit of chalk having physical dimensions.

As an aid in identifying various points, it is common to use dots and label the dots with capital letters for identification.

Line

Concept: A line has no height and no width, but it does have the property of infinite length.

Every place on the line has the property of position so every place on the line can be thought of as a point. A line is therefore an infinite set of points. A line extends infinitely in both directions and therefore has no endpoints.

Concept: Given two points, only one straight line can exist which contains both points, and there always is one such line.

Therefore, a straight line can be identified by any two points on that line. Those same two points cannot both be on any other straight line.

The symbol $\overleftrightarrow{AB}$ is used to indicate the line containing the points A and B.

Concept: A finite portion of a line is called a line segment.

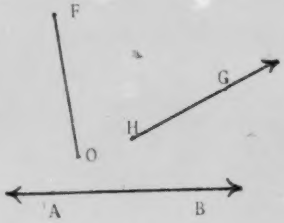
It has two endpoints. The line segment with endpoints A and B can be represented by the symbol $\overline{AB}$ which is read "line segment AB" or just "segment AB".

Concept: An infinite portion of a line which has only one endpoint is called a ray.

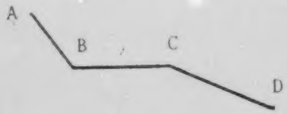
A ray is identified by the endpoint and any other point on it. If the endpoint is represented by the letter O, and the letter P represents any other point on the ray, the ray may be identified by the symbol $\overrightarrow{OP}$, which is read "ray OP." The first letter always represents the endpoint, which is also called the origin.

It is not necessary to use the symbols $\overleftrightarrow{AB}$, $\overline{OF}$, and $\overrightarrow{HG}$. It is just as correct to designate them as "line AB", "segment OF", and "ray HG."

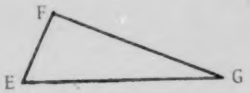
The diagram at the right represents $\overleftrightarrow{AB}$, $\overline{OF}$, and $\overrightarrow{HG}$.



A succession of connected straight line segments all of which do not lie on the same straight line is called a *broken line*. If the first and last line segments are connected to form a closed path, the figure is called a *closed broken line*.



broken line



closed broken line

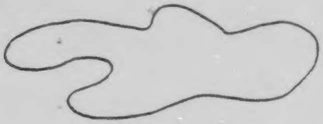
Thus we may speak of the broken line ABCD and the closed broken line EFG. The terms point, straight line, and plane should be left undefined. Attempts to define any of these terms usually result in definitions which are mathematically incorrect or far too complex for pupils at this grade level.

There are differences in methods used to teach the concept of a curve.

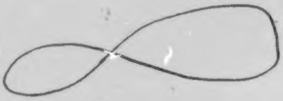
Concept: In higher mathematics all lines are considered as curves and a straight line as being a special type of curve.

There are texts which refer to broken line ABCD as curve ABCD and closed broken line EFG as closed curve EFG or simple closed curve EFG. (See the diagram on the previous page.)

A *simple closed curve* is one which does not intersect itself, and separates the plane into three sets of points: the curve, its exterior and its interior which is called a region.



simple closed curve



non-simple closed curve

Points that are contained on the same straight line or portion of a line are called *collinear points*. *Any two* points are collinear. Given any two points, a straight line can be drawn containing these two points. Lines which all contain the same point are called *concurrent lines*. Two lines which intersect are concurrent lines, and all concurrent lines intersect. Lines which are composed of the exact same set of points are called *coinciding lines*. When two lines are identified as being coinciding, this simply means that different symbols are being used to identify the same line.

Plane and Lines in a Plane

Concept: A plane has the properties of infinite length and infinite width, but it does not possess a third dimension.

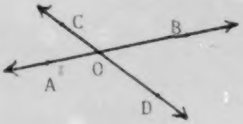
It has no thickness. It can be thought of as a perfectly flat surface. A plane can be identified by any three points not all on the same line, or by a straight line and a point not on that line.

A line divides a plane into two half-planes. All points are either on one side of the line, on the other side of the line, or on the line.

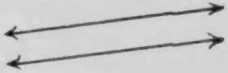
A plane is a set of points and it is also a set of lines. Every point in a plane can have an infinite number of lines going through it.

Points in the same plane are called *coplanar points*, and lines in the same plane are called *coplanar lines*.

Lines which contain a common point *intersect*. The point of intersection is on *each* of the intersecting lines. Two intersecting lines can be considered as forming four rays. In the diagram at the right, the intersection of lines AB and CD at point O form four rays, OA, OB, OC, and OD.



Concept: Parallel lines are straight lines in the same plane which do not intersect.



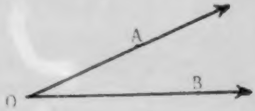
It is best to restrict discussion of parallel lines to straight lines only. [The question of whether or not there can exist parallel curves is a topic of higher mathematics and is far above the level of this course.]

Angle

Concept: An angle is the set of all points that are contained in two rays which have the same endpoint.

The rays are called the sides and the common endpoint is called the vertex.

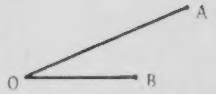
An angle is identified by the use of the letters that identify the rays, the three letters being arranged so that the letter that represents the *endpoint* is the *middle* letter. The angle in the diagram is identified as "angle AOB" or, by the use of the symbol ∠ , it may be identified as ∠ AOB. It may also be identified as "angle BOA" or "∠ BOA."



A concise method of identifying angles is by the use of a single letter or number at the vertex. This is acceptable, providing ambiguity does not result. The above angle may be identified simply as "angle O."

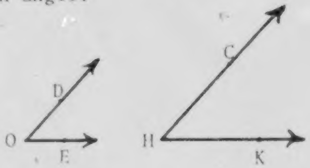
The set of all points contained in two line *segments* that have one common endpoint defines a part of an angle. The line segments are considered as parts of rays and the resulting figure is considered as a part of an angle. The use of line segments to form parts of angles is a very valuable tool in introductory geometry and no great harm results from referring to such a figure as an angle.

The figure at the right can be called "angle AOB" even though it technically is only a part of angle AOB formed by rays OA and OB, as in the previous diagram.



The sides of an angle are rays and so the sides are always of infinite length. However, the use of line segments to form angles sometimes leads pupils to believe that the length of the sides has something to do with the size of an angle. It must be clearly stated that the length of the line segments used to draw the angle has *no* relation whatsoever to the size of an angle.

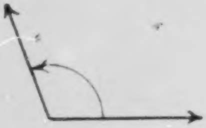
Angle DOE and angle CHK in the adjoining diagram have the same measure, with the second figure showing more of the rays that constitute the sides than the first figure does.



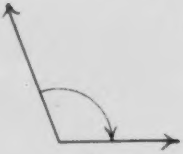
Concept: An angle is commonly measured in terms of counterclockwise rotation.

The amount of rotation necessary to rotate a ray completely about its endpoint so that it coincides with its original position constitutes one complete rotation. This amount of rotation is commonly divided into 360 equal units. Each such unit is called a *degree*. A degree is $\frac{1}{360}$ of a complete rotation; it is written as 1°.

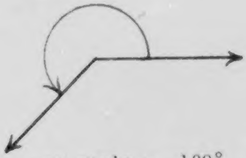
If an angle is ever measured in terms of clockwise rotation, this must be indicated. In cases where there is some ambiguity as to whether the measure of an angle is based on clockwise or counterclockwise rotation, a small arrow may be used to indicate direction of rotation. This is often used for angles of greater than 180° of rotation.



Counterclockwise rotation

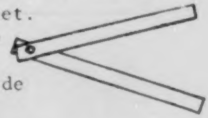


Clockwise rotation



an angle > 180°

A model of an angle may be made which consists of two cardboard strips fastened together at one end with an eyelet. The cardboard strips may be considered to be the sides of an angle, and the eyelet the vertex of the angle. Manipulation of the model will provide a means for achieving a better understanding.



Measurement of Angles

One way to think about the measurement of an angle is to consider it to be the measure of the amount of rotation that would be necessary to make the initial side of an angle coincide with the terminal side of that angle. Because positive angle measure as a measure of counterclockwise rotation is an important concept in mathematics, it may be well to emphasize the corresponding scale on a protractor. Usually a semicircular protractor has two scales. Let the pupils receive training first in the use of the scale for counterclockwise rotation, but since, at this level the direction of rotation is of minor importance, keep any discussion and emphasis on an informal level.

A clear plastic circular protractor can be used to measure angles of any size. However, only angles less than a straight angle can be measured directly with a semicircular protractor. To calculate the measure of a reflex angle by using a semicircular protractor, use the scale which is designed for measuring clockwise rotation. Subtracting the clockwise rotation from 360° will give the counterclockwise rotation. Again, since reflex angles should not receive too much emphasis at this level, this may be kept quite informal.

Angles of certain size and those that fall within a certain range of sizes have been given special names.

A *right angle* is an angle of 90° [or one-fourth of a complete rotation.]

A *straight angle* is an angle of 180° [or one-half of a complete rotation.] Two rays that form a straight angle constitute a straight line.

An *acute angle* is an angle less than 90°.

An *obtuse angle* is an angle greater than 90° but less than 180°.

A *reflex angle* is an angle greater than 180° but less than 360°.

If two angles have a common ray and a common vertex, and have no interior points in common, they are called *adjacent angles*.

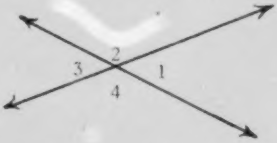
Angles as Applied to Lines and Planes

Concept: Complementary angles are two angles whose sum is equal to a right angle.

Concept: Supplementary angles are two angles whose sum is equal to a straight angle.

The two angles do not have to be adjacent. When two lines intersect, four pairs of supplementary angles are formed. The sum of the angles about a point on a plane is 360°.

The non-adjacent pairs of angles formed by two intersecting lines are called *vertical angles*. It is simple to show that vertical angles must be equal. In the diagram at the right $\angle 1 + \angle 2 = 180^\circ$; $\angle 3 + \angle 2 = 180^\circ$; $\angle 1 + \angle 2 = \angle 3 + \angle 2$; $\angle 1 = \angle 3$.



The other vertical angles can be shown to be equal by use of this same procedure.

Concept: Two lines are perpendicular if they intersect to form right angles.

If they intersect at any other angle, they are *oblique* to each other.

The definition of horizontal and vertical lines (and planes) is usually far more technical and complicated than most teachers realize. It is best to begin be defining a vertical line. To do this, suspend a weight at the end of a string so that the weight hangs freely. The line in which this string hangs is called a plumb line. A plumb line is a *vertical line*. A plane which contains a vertical line is called a vertical plane.

The word "horizontal" has little real meaning unless you answer the question *"Horizontal with reference to what?"*. In most instances, horizontal is used to refer to a plane being perpendicular to a vertical line within the classroom.

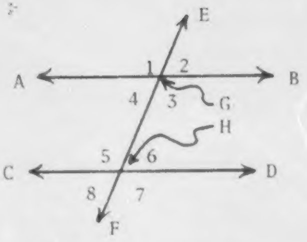
A horizontal line is a line perpendicular to a *particular* vertical line or plane.

When drawing lines on a piece of paper, it is very common to refer to the lines going from left to right as horizontal, and those running from the top of the paper to the bottom as being vertical. There is nothing wrong with such usage and the model is very convenient, but it may be pointed out that, while the paper is on the top of the desk, both sets of lines are horizontal. They are all in the same plane. When the paper is posted on the bulletin board, those lines drawn from left to right can then be considered as being horizontal and those drawn from the top to the bottom can be considered as being vertical.

The topic of angles may be explored further in the eighth grade. The following angle relationships are introduced on this level using parallel lines and a transversal. A diagram, with the angles numbered for easy reference and used to show the pairs of angles, will be much less confusing than strict verbal definitions.

Draw line AB is parallel to line CD. Transversal EF intersects AB at G and CD at H.

When two parallel lines are cut by a transversal pairs of *corresponding angles* and *alternate interior angles* are formed.



Corresponding angles - These are pairs of equal angles as

∠ 1 and ∠ 5	∠ 2 and ∠ 6
∠ 4 and ∠ 8	∠ 3 and ∠ 7

Alternate interior angles - These are pairs of equal angles as

∠ 4 and ∠ 6
∠ 3 and ∠ 5

In the preceding diagram if we are given one of the eight angles in degree measure, the other seven may be found by a number of methods.

i.e. If ∠ 1 = 110°

∠ 1 + ∠ 2 = straight angle = 180°

By substitution, the value of ∠ 2 = 70°

Now ∠ 1 = ∠ 3 and ∠ 2 = ∠ 4 since they are pairs of vertical angles.

Hence ∠ 3 = 110° and ∠ 4 = 70°

Also, ∠ 3 = ∠ 5 and ∠ 4 = ∠ 6 since they are pairs of alternate interior angles.

Hence ∠ 5 = 110° and ∠ 6 = 70°

Finally, ∠ 5 = ∠ 7 and ∠ 6 = ∠ 8 since they are pairs of vertical angles.

Hence ∠ 7 = 110° and ∠ 8 = 70°

Polygon

Concept: A polygon is a closed broken line in a plane.

Almost all are named according to the number of their angles. The quadrilateral is named for its number of sides.

Three angles:	triangle	Eight angles:	octagon
Four sides:	quadrilateral	Nine angles:	nonagon
Five angles:	pentagon	Ten angles:	decagon
Six angles:	hexagon	Twelve angles:	dodecagon
Seven angles:	heptagon	Twenty angles:	icosagon

In a polygon, the intersection of two sides is the *vertex*, and the line segment joining two non-adjacent vertices is a *diagonal*.

Concept: A regular polygon is a polygon that is both equilateral and equiangular.

It is possible for a polygon to be equilateral without being equiangular, for example, a rhombus; and to be equiangular without being equilateral, for example, a rectangle.

Triangle

A triangle can be classified according to the type of angles it contains. An *acute* triangle contains only acute angles. A *right* triangle contains a right angle. An *obtuse* triangle contains an obtuse angle. An *equiangular* triangle contains three 60° angles.

A triangle can also be classified according to the number of equal sides it has. An *isosceles* triangle has two equal sides and an *equilateral* triangle has three sides equal. If its sides are unequal, the triangle is *scalene*.

Concept: An altitude of a triangle is the perpendicular line segment joining a vertex of the triangle to the line that contains the opposite side.

The word "altitude" has three distinct meanings.

(1) The line segment described above
(2) The length of the line segment described above
(3) The line containing the line segment described above

When the word "altitude" is used, it must be clear from the context which of these three meanings is intended.

Every triangle has three altitudes. At this point, the pupils have not yet learned to construct perpendiculars so it probably is best to have them use a cardboard or some other device with a right angle to draw in the perpendicular lines that are the altitudes. It is sometimes difficult for students to understand that the altitudes may be *outside* the triangle and to construct or draw them it is necessary to extend the sides. The use of the word "height" may be helpful in teaching one concept of altitude, as the pupil can see that the altitude may be the vertical height of the vertex above the base.

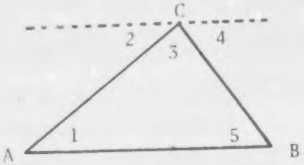
The use of the word height also makes the use of the letter "h" in the area equation $A = \frac{1}{2}bh$ seem logical.

Concept: A median of a triangle is the line segment drawn from a vertex to the midpoint of the opposite side.

Finding the midpoint of the opposite side may pose a problem in that pupils have not learned to determine the midpoint of a line segment. This can be done by either measuring or by folding the paper upon which the triangle has been drawn. The medians meet at a single point two-thirds the distance from any vertex to the opposite side. This can be presented just as an interesting fact.

Concept: The sum of the angles of a triangle is equal to 180°.

This is an important concept. The easiest way of demonstrating this is to tear off the three angles of a triangle and fit them together to show they form a straight angle. A second way is to fold the angles so that their vertices meet on the base and they will form a straight angle. If the fact that alternate interior angles of parallel lines cut by a transversal are equal has been mastered, a more formal proof may be used. This consists of drawing a line through vertex C parallel to the base of the triangle. ∠1 = ∠ 2 and ∠ 5 = ∠ 4 because alternate interior angles of parallel lines intersected by a transversal are equal. ∠ 1 + ∠ 3 + ∠ 5 = ∠ 2 + ∠ 3 + ∠ 4. ∠ 1 + ∠ 3 + ∠ 5 equals a straight angle because ∠2 + ∠3 + ∠4 equals one straight angle. Therefore, the sum of the angles of a triangle is equal to 180°.



Concept: The sum of the lengths of any two sides of a triangle must always exceed the length of the third side.

This can be demonstrated by having the pupils attempt to draw or construct with physical objects a triangle in which the sum of the lengths of two given sides does not exceed the length of the third side. The pupils will find the construction impossible under these conditions and thus will see the need of the restriction given above.

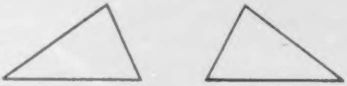
Concept: The triangle is the only rigid polygon.

The shape of a triangular figure or object cannot be changed without changing the length of one or more sides. This characteristic rigidity of triangular figures and objects is the reason for use of triangular forms of various kinds in construction and engineering work.

This characteristic also may find use in the concept of congruency of triangles.

Concept: If two triangles have their corresponding sides and angles respectively equal, they are called congruent.

Congruent triangles can be made to coincide. In making the two triangles coincide, any type of rigid motion through space is allowed. For example, the triangles in the accompanying diagram are congruent, even though rotation through three-dimensional space is necessary to make them coincide. Conversely, as long as the triangles can be made to coincide, they are congruent.



Concept: If three sides of one triangle are equal respectively to each of the three sides of another triangle, the two triangles can be made to coincide and are congruent.

This is actually a postulate, but it is a conclusion easily arrived at through a discussion of the rigidity of a triangle and through physical manipulation and comparison of various congruent triangles.

Concept: Corresponding angles of congruent triangles are equal.

This follows from the definition of congruence but can be arrived at also through physical comparison of congruent triangles.

Quadrilaterals

The depth to which this topic is treated will depend upon the mathematical ability of the pupils. It is not intended that pupils merely memorize these definitions but that considerable experimentation be included.

A trapezoid is a quadrilateral which has only one pair of opposite sides parallel. A parallelogram is a quadrilateral which has both pairs of opposite sides parallel.

The opposite sides and opposite angles of a parallelogram are equal. This can be demonstrated by having the students draw several parallelograms, cut them in half and then place one half over the second half, hold up to a source of light, and actually observe this relationship. In a like manner, the fact that the diagonals of a parallelogram bisect each other can be demonstrated by folding the parallelogram at the point of intersection of the diagonals so that the crease is at right angles to one diagonal and then repeating for the other.

A rectangle is a parallelogram whose angles are right angles. To be more mathematically precise, the rectangle can be defined as a parallelogram one of whose angles is a right angle.

A rhombus is a parallelogram whose sides are equal. To be more mathematically precise,the rhombus can be defined as a parallelogram having two adjacent sides equal.

A square is a rectangle whose sides are equal. A square can also be defined as a rhombus whose angles are right angles. If desired, the more precise definition of a square as being a rectangle having two adjacent sides equal may be used. Because the opposite sides of any parallelogram are equal, if two adjacent sides are equal, then all the sides must be equal.

Circle

Concept: A circle is the set of all points in a plane which are at a fixed distance from a fixed point.

The fixed distance is called the radius and the fixed point is called the center. It is very important to include the words "in a plane" in the definition of a circle. Otherwise, the definition will be that of a sphere.

All points whose distance from the center is less than the radius are said to be in the interior of the circle, and the set of all such points plus the center is the interior of the circle. All points whose distance from the center is greater than the radius are said to be exterior to the circle and the set of all such points is the exterior of the circle. A circle divides a plane into three non-intersecting sets of points, (that is, sets which have no points in common) the interior, the exterior and the circle itself. Any point in the interior or any point in the exterior is not on the circle. This is contrary to much of the language commonly used in reference to circles. Very often when we refer to a circle, we really refer to the circle plus its interior.

Concept: The circle plus its interior forms a closed circular region.

For example, reference is often made to the area of a circle. A circle is a curved line. It has no dimensions except length. It has no area. It is the *region* which is composed of the circle and its interior which has area. No doubt the phrase "area of a circle" will continue to be used, but in using it, the pupil should be fully aware of the more complete mathematical description which is being abbreviated in this usage.

The same applies to polygons. A polygon is composed of various line segments. It has no area. It is the region which is composed of the polygon plus its interior which has area.

A chord is a straight line segment whose endpoints are points on the circle. A chord which contains as one of its points the center of the circle is called a diameter.

The radius of a circle is the distance from the center to the circle. It is also a line segment whose endpoints are the center and a point on the circle. The word "radius" is used for both the distance and the line segment.

A semicircle is half a circle. It should be represented by an arc as shown in figure A in the adjoining diagram and not by figure B the region shown in this diagram. However, the latter will no doubt continue to be referred to as a semicircle, although it should be called a semicircular region.

figure A

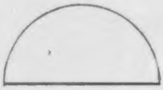


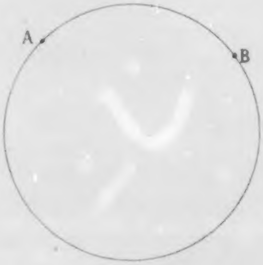
figure B

The topic of circles may be extended further in the eighth grade with several additional concepts.

Concentric circles are circles in the same plane which have the same point as their center but have radii unequal in length.

A central angle is an angle whose vertex is the center of the circle. Its sides are rays, each containing a radius of the circle.

An arc of a circle is the set of all points on a circle between two given points on the circle, including the given points. An arc cannot be identified without ambiguity by just the two given points, as there is no way of knowing which set of points on the circle is being referred to. Arc AB could refer to either the top or the bottom part of the circle. Three letters representing respectively an endpoint, a point on the arc, and its other endpoint, identify an arc exactly. However, it is customary to name a *minor* arc by its two endpoints only and this is proper if the convention is agreed upon.



Thus it follows that a minor arc $< 180°$, and a major arc $> 180°$.

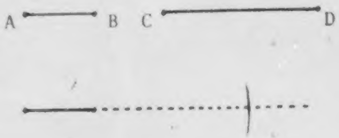
Measurement and Constructions

The constructions in this unit are to be performed with the use of a straight edge and pair of compasses only.

The compasses are used to determine the positions of a pair of points or a set of points. The straight edge is used to determine the position of points on the line identified by two given points.

The pupil should be shown how to use a pair of compasses to mark off two points a distance apart equal to the distance between two given points, and then how to construct a line segment equal in length to a given line segment by marking off two points the same distance apart as the endpoints of the given line segment and connecting these points with a line segment.

The next step is to construct a line segment equal in length to the sum of the lengths of two given line segments. To construct a line segment equal in length to the sum of the length of given segments AB and CD, first construct a segment equal in length to AB. The problem now is to mark off the distance equal to the length of CD from the end of the segment just constructed.

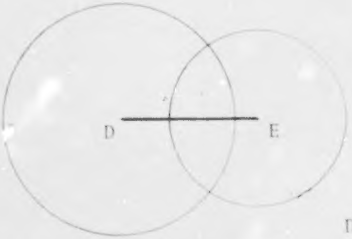


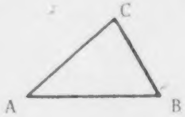
The compasses are used to mark off the distance equal to the length of CD but there is doubt as to exactly where the endpoint of the second segment is to be located. The pupil knows the region where it must be located but does not know its exact position. This can be indicated by constructing in the region where the endpoint of the second segment must be located a small arc containing many points at the distance equal to the length of CD from the end of the first segment. The first segment is then extended to intersect the arc. If preferred, a line may first be drawn and then segments equal in length to AB and CD marked off, starting at a point arbitrarily fixed on the line.

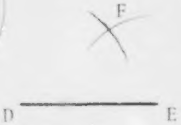
The use of an arc to indicate the set of all points a fixed distance from a fixed point within a certain region is extremely useful in the majority of the constructions to be performed. When the location of one endpoint of a line segment to be constructed is known and the length of the segment is known, the other endpoint must be located on a circle whose center is the known endpoint and whose radius is the length of the segment. If it is known that the second endpoint must be located within a certain region, it is not necessary to construct the entire circle. Only that part of the circle within that region is necessary.

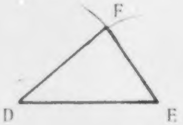
It may be best to have the student first construct the entire circle and lead him to the realization that constructing the entire circle is not necessary when the location of the second endpoint is known to be within a given region.

To construct a triangle congruent to a given triangle or to construct a triangle given the three sides, the location of three points must be determined. When a line segment is constructed equal in length to one of the sides of the triangle or equal in length to one of the given line segments, its endpoints constitute two of the necessary three points. The problem is simply to determine the third point.





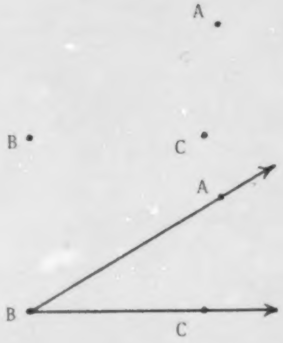




Given triangle ABC in the above diagram, DE is constructed equal in length to AB. Point F will be located the distance equal to the length of AC from point D and the distance equal to the length of BC from point E. Point F must be located on a circle whose center is D and radius is equal to the length of AC and it must also be a point on the circle whose center is E and whose radius is equal to the length of BC. When these two circles are constructed, their intersection will be the point F. The intersection of the two circles will actually result in two points. Connecting these points to the segment DE will result in two triangles, each congruent to the given triangle. Since only one triangle is desired, it is only necessary to construct arcs in the one region where it is desired that point F will be located.

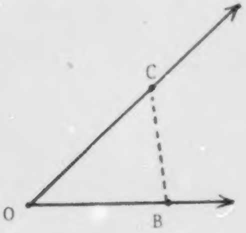
If the pupil can construct a triangle congruent to a given triangle and if he knows that corresponding angles of congruent triangles have equal measure, he is ready to learn how to construct an angle equal in measure to a given angle.

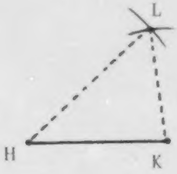
An angle is determined by three points, one point on each ray and the endpoint common to each ray. The point which is the endpoint must be indicated. If it is known that points A, B, and C, are on a certain angle, and that point B is the vertex, and that A and C are on different rays, the angle can be constructed by drawing a ray from point B through point A and a ray from point B through point C.

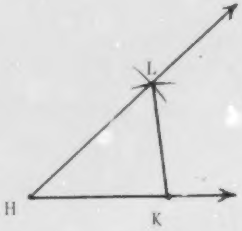


Three points also determine a triangle. The three given points can be considered as being the three vertices of a triangle.

In the accompanying diagram, let B and C be fixed points on the sides of angle BOC. To construct an angle KHL equal in measure to given angle BOC, construct HK equal in length to OB. Point H will be the vertex of the angle being constructed. Point L will be at the distance equal to the length of OC from point H and at the distance equal to the length of BC from point K.

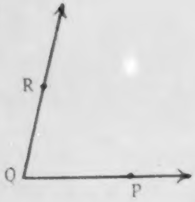


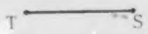


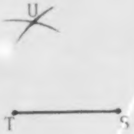


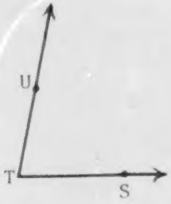
Drawing in the line segments produces a triangle congruent to triangle BOC. Corresponding angles of congruent triangles have equal measure so angle KHL is equal in measure to angle BOC. Extending the sides HK and HL produces the rays that correspond to rays OB and OC.

Once the fact has been established that this procedure results in the construction of an angle equal in measure to a given angle, the procedure may be simplified in that it is not necessary to draw in segment LK. To construct an angle equal in measure to a given angle PQR, mark off S and T so that the length of ST = length of PQ, and then determine point U so that length of TU = length of QR and length of SU = length of PR. Once point U has been determined, draw rays from point T through S and from point T through point U.

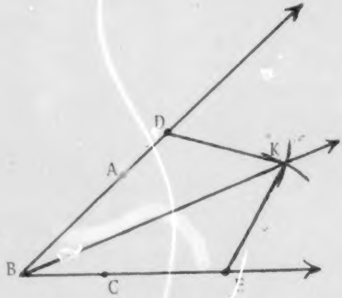






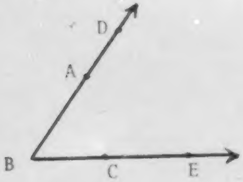


An angle is bisected by a ray if the ray is in the interior of the angle and if the ray forms two angles of equal measure within the angle. The method of bisecting an angle involves the construction of two congruent triangles within the angle. Extending the side common to both triangles results in a ray within the original angle which forms two angles equal in measure with the sides of the original angle.

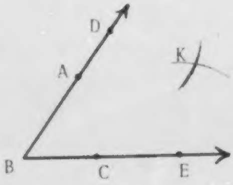


Given angle ABC, determine a point E on ray BC and a point D on ray BA so that BE is equal in length to BD. Determine a point K which is equidistant from the two points D and E. Draw in segments DK and EK, and BK. Extend BK to indicate ray BK.

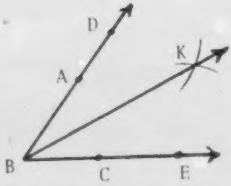
The three sides of triangle BKD are equal respectively in length to the three sides of triangle BKE so the two triangles are therefore congruent. Angle CBK is equal in measure to angle ABK because they are corresponding angles of congruent triangles.



step 1



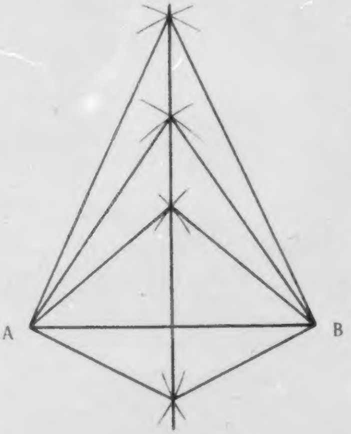
step 2



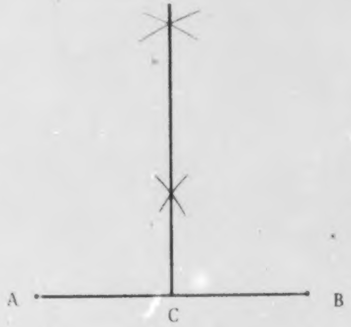
step 3

Ray BK is in the interior of the angle and it forms two angles of equal measure with the sides of angle ABC. It, therefore, bisects angle ABC. The procedure may be simplified as in the preceding three step diagram since it is not necessary to draw in segments DK and EK.

The method of determining the midpoint of a line segment by construction is arrived at intuitively by constructing several points equidistant from the endpoints of the line segment and observing that all such points are on the same line. By constructing two such points, joining them with a line segment and extending the segment until it intersects the given line segment, the midpoint of the given line segment is determined.

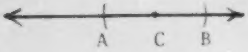


The construction can then be simplified by not drawing in the segments from the endpoints of the given line segment. See diagram at the right.

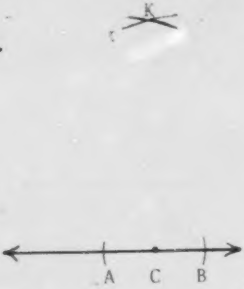


Erecting a perpendicular at a given point on a line consists of bisecting a 180° angle. A perpendicular to a line forms a 90° angle with the line. If it forms one 90° angle, it must form two 90° angles because the two angles must be supplementary and the supplement of 90° is 90°.

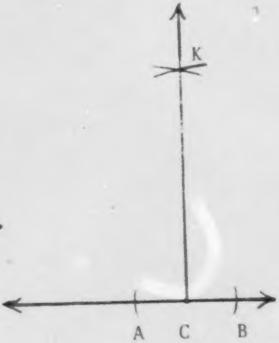
The given point on the line may be considered as being the vertex of a 180° angle and the two parts of the line may be considered as being the rays forming its sides. To construct two 90° angles at this vertex, simply bisect the 180° angle in accordance with previous instructions for bisecting angles. These steps are shown in the accompanying diagram.



step 1



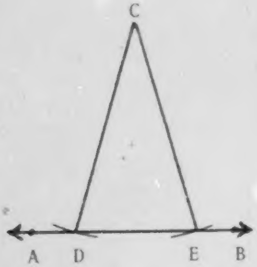
step 2

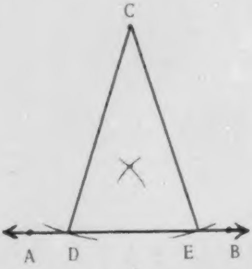


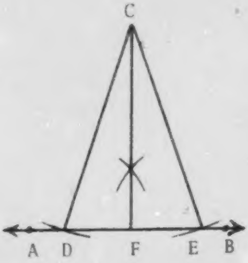
step 3

As shown in the three diagrams below, the construction of a perpendicular from a point to a line consists of constructing two congruent triangles, the three sides of one triangle being equal respectively in length to the three sides of the second triangle. Measure of ∠EFC = measure of ∠DFC because corresponding angles of congruent triangles have equal measure. These angles of equal measure are also supplementary so they must both be right angles. Therefore, the segment CF is a perpendicular from point C to line AB.

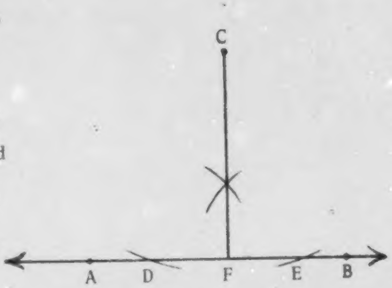
First construct two equal segments from point C to line AB, intersecting AB at points D and E. Bisect DE. The two triangles formed are congruent because length of CD = length of CE, length of DF = length of EF, and length of CF = length of CF. ∠EFC and ∠DFC are equal in measure and supplementary, and are, therefore, right angles. CF is perpendicular to AB.







The construction may be simplified by not drawing in segments CD and CE. See diagram at the right.



Since two points determine a line, CF is determined by point C and the intersection of the second set of arcs. If these two arcs intersect very near C, it will be difficult to draw CF accurately. Thus, for accuracy, it is well to have these arcs intersect closer to line AB than to point C, or even on the other side of AB.

Measurement of Distances

Concept: The distance between two points, unless otherwise indicated, is always interpreted to be the length of the straight line segment whose endpoints are these two points. The shortest distance between two points is the straight line distance.

At this time it may be well to discuss with pupils the fact that the distance between two points on the surface of the earth is usually interpreted as meaning the distance along the surface of the earth. This is *not* the straight line distance. The shortest distance between two points on a sphere (the earth is not a perfect sphere) is along the great circle route, a great circle being the set of all points on a sphere contained on one plane which has as one of its points the center of the sphere. On the other hand it is true that for relatively short distances on a sphere, the straight line distance serves as a good approximation for the arc distance.

The shortest distance between two parallel lines, parallel planes, a point and a line, and a point and a plane is always the perpendicular distance. This conclusion can be arrived at intuitively.

The topic of measurement of distances should include finding the perimeter of any polygon. Rather than using specialized formulas for perimeters of figures, the concept of the sum of the lengths of the sides of the polygon should be stressed. Students will be able to formulate their own perimeter notation.

The topic of measurement of distances should include the measurement of the lengths of diameters and circumferences of circles. The circumference of a circle means the length of the circle (the circle is the curved line itself). The length of the diameter of a circle can be measured easily enough, but measuring the circumference which is the length of a curved line, poses a problem. One method of measuring the circumference is to cut the region bounded by the circle from heavy paper or cardboard and measure the circumference with a cloth tape or flexible steel tape. The teacher may have to use some ingenuity in helping pupils devise ways of measuring circumferences. The use of tin cans may be **helpful.** The measurement of the circumferences are then compared with the measurement of the lengths of the diameters. Allow the pupils to discover the relationship. At first, they should be able to see that each circumference is a little over three times the length of the diameter. With more precise measurements, the circumference should always be fairly close to 3.14 times the length of the diameter.

Concept: The ratio of the circumference of a circle to the length of its diameter is a constant. This number is called pi and its symbol is π. This is the only correct symbol for that number. It is an irrational number.

Such irrational numbers do not have much meaning to pupils unless they are compared to some rational number. Performing numerical calculations involving pi involves converting pi to some rational number, *approximately* equal to pi, if the answer is required to be a rational number. However, if a rational answer is not required, it is not necessary that pi be approximated. A circle with a diameter of 3 inches has a circumference of 3π inches. Pi is a real number. A line can be 3π inches long. However, we usually prefer the use of rational number as answers so pi is approximated as $3\frac{1}{7}$, 3.14 or to as many decimal places as desired. Approximations of pi have been calculated to 500,000 decimal places, that is, the approximations and pi differ by an exceedingly small number which has zero's in at least the first 500,000 decimal places.

This extreme accuracy is of theoretical importance only, for practical purposes 2, 3 or 4 places suffice.

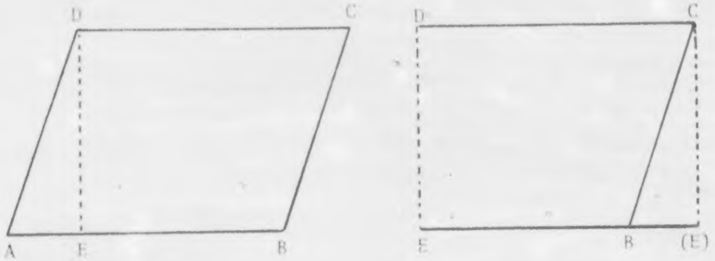
Measurement of Areas

The method of determining the area of a rectangle is arrived at intuitively. Have the pupils cut out 20 or 30 one-inch squares and define the area of a one-inch square as one square inch. By determining how many of these squares will cover the surface of various rectangles, the formula for the area of a rectangle can be formulated by the pupils. Next, have the pupils make a supply of half-inch squares with an area of one-fourth of a square inch. They can then determine the area of rectangles whose dimensions contain the fraction one-half. This will verify the formula for the area of the rectangle they previously determined.

If the pupil will draw several parallelograms with the same base but with different altitudes, he can arrive at the conclusion that the area of the parallelogram is very much dependent on the length of the altitude. By constructing several parallelograms with the same altitudes but different bases, he can see how the area is dependent on the length of the base. The formula for the area of a parallelogram is derived by converting any parallelogram to a rectangle which has an area equal to that of the parallelogram.

As shown in the accompanying diagram, any parallelogram ABCD can be converted to a rectangle by dropping a perpendicular from D to AB, cutting off triangle AED and then placing triangle AED along CB so that AD will coincide with CB. The resulting figure is seen to be a rectangle because the opposite sides are still equal and the angles are all right angles. In general, an equivalent rectangle can be obtained by dropping any perpendicular cutting *both* bases and interchanging the positions of the two sections formed. Good pupils may wish to investigate very oblique parallelograms in which no perpendicular cuts both bases.

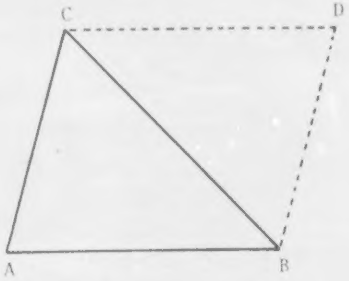
The length of the base and the length of the altitude of the original figure have not been changed. The area of the rectangle is equal to the area of the original parallelogram. The area of the rectangle is found by the equation $A = lw$. In the case of the parallelogram, its base corresponds to the length of the rectangle and the length of its altitude corresponds to the width of the rectangle. Therefore the area of the parallelogram can be calculated by use of the equation $A = bh$, where "b" is the length of the base and "h" is the length of the altitude of the parallelogram.



Again, this and the following developments should be kept very informal with emphasis on the discovering of the relations through the use of models, cut-outs, and other concrete materials, much of which can be pupil made.

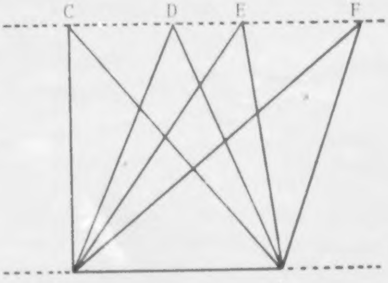
The formula for the area of a triangle may be derived from the concept that any two congruent triangles can be arranged to form a parallelogram. Therefore, the area of a triangle is one-half the area of a parallelogram with the base and altitude equal to the length of the base and altitude of the triangle.

Given triangle ABC, in the accompanying diagram, draw CD parallel and equal to AB, and draw BD, forming another triangle CBD and the parallelogram ABDC. The two triangles formed are congruent (the three sides of one are equal respectively to the three sides of the other) and can be made to coincide, so their areas are equal. The area of the parallelogram can be calculated by the use of the formula $A = bh$, so the area of the original triangle can be calculated by use of the formula $A = \frac{1}{2}bh$.

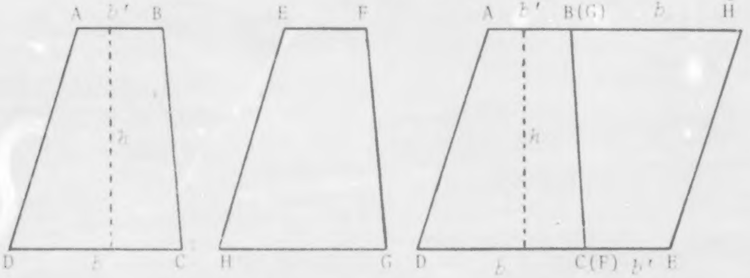


Concept: The area of a triangle depends only on the lengths of the base and the altitude. It may be unchanged by changes in the lengths of the other two sides. All triangles with the same base and equal altitudes have equal areas.

Triangles ABC, ABD, ABE, and ABF all have equal areas because they have the same base and equal altitudes. A model in which AB is ruled line segment on a sheet of cardboard and AC and BC are rubber bands is useful in showing various possibilities.

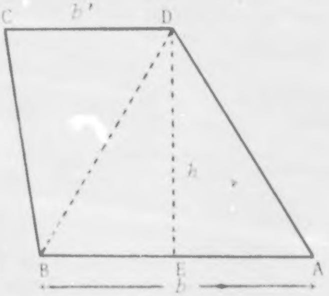


In the eighth grade, the area of a trapezoid may be considered. The formula for the area of a trapezoid can easily be derived from that of a parallelogram by considering two congruent trapezoids which are placed together after one has been rotated through 180°. Thus, in the accompanying diagram, ABCD and EFGH are two congruent trapezoids which are placed together after one has been rotated through 180°. Thus, in the accompanying diagram, ABCD and EFGH are two congruent trapezoids with bases b and b and altitude h. Trapezoid EFGH is rotated and placed adjacent to trapezoid ABCD so that FG coincides with BC. A parallelogram is formed with base (b+b') and altitude h. Its area is h(b+b'). Therefore, the area of the trapezoid ABCD is $\frac{1}{2}h(b+b')$.



An alternate way in which the formula for the area of a trapezoid could be developed involves the use of triangles and the formula for the area of a triangle, $A=\frac{1}{2}bh$.

Any trapezoid may be divided into two triangles by drawing a diagonal. In trapezoid ABCD at the right, drawing diagonal DB forms triangle ABD and triangle DBC. Altitude DE drawn from D perpendicular to AB is equal to the altitude from B to side DC. The two triangles have equal altitudes. The area of the trapezoid is equal to the sum of the areas of the two triangles. Thus, $A = \frac{1}{2}bh + \frac{1}{2}b'h = \frac{1}{2}h(b+b')$ as before.



The formula for the area of a circle, $A = \pi r^2$, may be introduced. Again, it involves the number pi, which must be approximated if a rational answer is required. Otherwise, pi is not replaced.

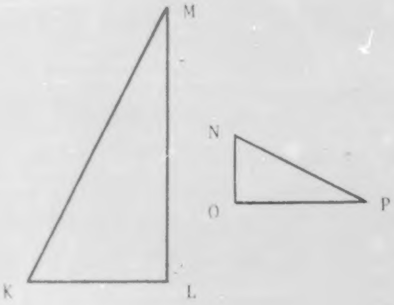
Proportions as Applied to Similar Triangles

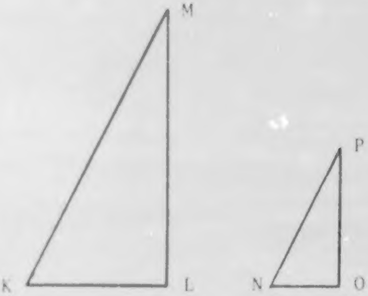
If the three angles of one triangle are equal respectively to the three angles of another triangle, the two triangles are similar.

Concept: If two triangles are similar, one is like a scale drawing of the other. The lengths of the sides of one of the triangles are proportional to the lengths of the sides of the other triangle.

The ratio of the length of one side of one of the triangles to the length of the corresponding side of the other triangle is equal to the ratio of either of the other sides of the first triangle to the length of the corresponding side of the second triangle. Any such pair of equal ratios can be expressed as a proportion. The corresponding sides are therefore said to be proportional. The ratio of the lengths of any two sides of one triangle equals the ratio of the lengths of the corresponding sides of the other triangle, taken in the same order.

As an aid in determining which sides of two triangles are corresponding sides, one triangle may be rotated about a vertex or revolved about a side until the angles of the second triangle are arranged in like manner to those of the first triangle. For example, given triangle KLM and triangle NOP, to determine which are the corresponding sides, triangle NOP is rotated so that its angles are arranged in the same order as those of triangle KLM. (See diagrams at the right.)

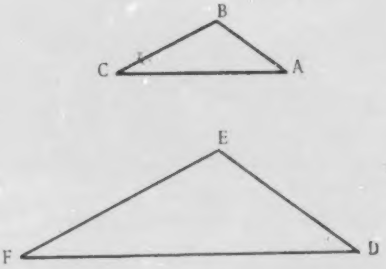




Indicate the three pair of corresponding sides in triangles KLM and NOP.

Answer: KL and NO are corresponding sides. LM and OP are corresponding sides. KM and NP are corresponding sides.

Given that triangles ABC and DEF are similar triangles, write as many different proportions as possible, applying the principle that corresponding sides of similar triangles are proportional.



Answer:

AB:DE = BC:EF
AB:DE = AC:DF
AB:BC = DE:EF
AB:AC = DE:DF
BC:EF = AC:DF
BC:AC = EF:DF
BC:AB = EF:DE
BC:EF = AB:DE
DE:AB = DF:AC
DE:EF = AB:BC
DE:DF = AB:AC
DE:AB = EF:BC

EF:BC = DE:AB
EF:BC = DF:AC
EF:DE = BC:AB
EF:DF = BC:AC
DF:AC = DE:AB
DF:AC = EF:BC
DF:DE = AC:AB
DF:EF = AC:BC
AC:DF = AB:DE
AC:AB = DF:DE
AC:BC = DF:EF
AC:DF = BC:EF

Answer the following questions.

(a) If triangle ABC and DEF are similar triangles in which AB = 8, AC = 14, DE = 12, and EF = 15, write a proportion that can be used to determine the length of BC, and solve this proportion for BC.

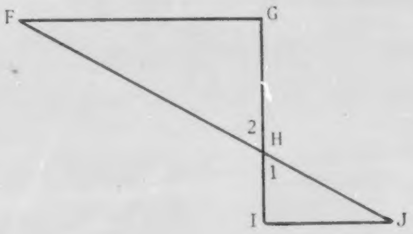
(b) Find the length of DF by use of a proportion.

Answers:

(a) AB:DE = BC:EF
8:12 = x:15
120 = 12x
10 = x
BC = 10

(b) AB:DE = AC:DF
8:12 = 14:x
8x = 168
x = 21
DF = 21

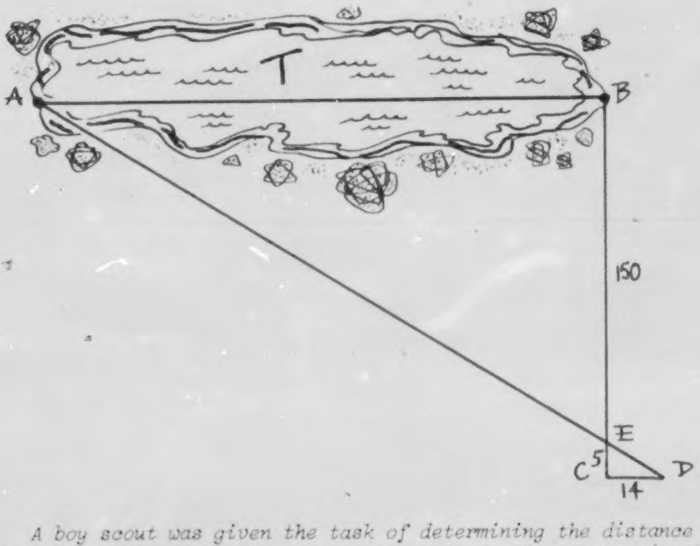
In the diagram at the right, angle FGH and angle HIJ are right angles and FJ and GI are straight lines. Is right triangle FGH similar to right triangle HIK?



Answer: Angle 1 and angle 2 are equal because they are vertical angles and vertical angles are equal. Two right triangles are similar if an acute angle of one triangle equals an acute angle of the other triangle.

Identify the three pairs of corresponding sides of the similar triangles in the previous question.

Answer: Sides FG and IJ are corresponding sides.
Sides GH and HI are corresponding sides.
Sides FH and HJ are corresponding sides.



A boy scout was given the task of determining the distance across Lake T shown in the diagram above from point A to point B. He first paced off an arbitrary distance of 155 feet from B to C at right angles to line segment AB. He then paced off an arbitrary *distance of 14 feet from C to D at right angles to BC. With the aid of a homemade transit and a fellow boy scout, he sighted from point D to point A and determined point E. By measuring, he found BE to be 150 feet and EC to be 5 feet.*

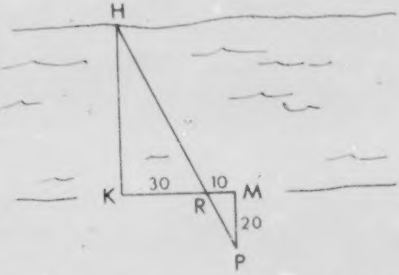
(a) Are triangles ABE and DCE each a right triangle?
(b) Does angle CED equal angle AEB?
(c) Are triangle ABE and triangle DCE similar triangles?
(d) Identify the three pairs of corresponding sides of triangle ABE and triangle DCE.
(e) Write a proportion that can be used to find the length of AB and solve the proportion for AB.

Answers:

(a) Triangle ABE is a right triangle because he marked off BC at right angle to AB. Triangle DCE is a right triangle because he marked off CD at a right angle to BC.
(b) Angle CED and angle AEB are equal because they are vertical angles and vertical angles are equal.
(c) Triangle ABE and triangle DCE are similar triangles. Any two right triangles are similar if an acute angle of one of the triangles equals an acute angle of the other triangle.
(d) Sides BE and EC are corresponding sides
Sides AB and CD are corresponding sides
Sides AE and ED are corresponding sides
(e) AB:CD = BE:EC
x:14 = 150:5
5x = 2100
x = 420

Lake T is 420 feet from point A to point B.

A second boy scout was assigned the task of determining the distance across Trout Creek from point H to point K. He paced off



an arbitrary distance of 40 feet from K to M at right angles to HK. He paced off an arbitrary distance of 20 feet from M to P at right angles to KM. With the aid of a homemade transit and a fellow boy *scout, he sighted from point P to H and determined point R. By measuring, he determined KR to be 30 feet and RM to be 10 feet.*

(a) Are triangles HKR and PMR similar triangles?

(b) Write a proportion that can be used to determine the length of HK and solve the proportion for HK.

Answers:

(a) Triangles HKR and PMR are similar triangles. Angle PRM is equal to angle KRH because they are vertical angles and vertical angles are equal. Triangles HKR and PMR are right triangles and right triangles are similar if an acute angle of one of the triangles is equal to an acute angle of the other triangle.

(b) RM:KR = MP:HK

10:30 = 20:x

10x = 600

x = 60

Trout Creek is 60 feet wide from point H to point K.

Polyhedrons

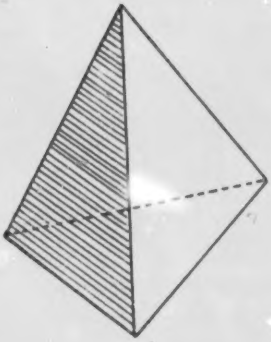
In the eighth grade, some of the informal ideas about solids may now be organized and extended. The purpose of this section is to teach the concept of a solid, a polyhedron, regular and irregular polyhedrons, the method of classifying polyhedrons, and definitions of vertices, faces, and edges of polyhedrons.

Before the work in this unit is introduced, pertinent topics from the previous year's work should be reviewed including the concepts of point, line, plane, line segment, polygon, names of common polygons, regular polygon, congruent, and definitions of common polygons.

The topic of polyhedrons is introduced by first discussing what is meant by a solid. A geometric solid may be something quite different from what the pupil may have been used to calling a solid. This concept of a geometric solid should be introduced carefully in simple logical steps beginning with what the pupil already understands. During the previous year, the pupil has been introduced to the concepts of point, line, plane, and line segment. The topic of solid can be introduced first through the use of these terms.

The questions and activities begin with a discussion of the simplest set of points, other than the null set. This is the set which consists of a single point. The next set discussed is the set consisting of two different points. Any two points identify a line and they identify the set of points contained on that line. Two points also identify the line segment whose end points are the two given points. The next set of points taken up is the set of three points not all on the same line. Such a set of points identifies a plane and the points contained on that plane. These are sets with which the pupils are already familiar.

The next set is the set of four points, no three of them contained on the same line and not all of them contained on the same plane. Such a set of points identifies a solid.



When each of four given points contained in such a set is connected to each of the other points, four triangles are formed, and each point is the vertex of three triangles. Each of the triangles is in a different plane.

The portion of any plane contained in the region bounded by a triangle is called a triangular region. The four given points in the above set form four triangular regions. These triangular regions, called faces, are so connected that they form a closed surface which completely encloses a portion of space.

Concept: A solid is any closed surface which completely encloses a portion of space.

Notice that a solid is a surface. This concept of a geometric solid may be quite different from the concept of a physical solid which most pupils have. A geometric solid has area but it does not have volume. When we refer to the volume of a solid, we are actually referring to the volume of the interior of the solid.

This can be compared to what is meant by the area of plane figures such as a circle. A circle is the set of all points in a plane a given distance from a given point. A circle is a curved line. It has no width or height, only length. It therefore has no area. When we speak of the area of a circle, we actually are referring to the area of the region bounded by the circle.

Some solids are closed surfaces formed by portions of the planes bounded by line segments joining four or more points not all on the same plane. The given points are called the vertices of the solid.

The line segments joining any two of the given points are called the edges of the solid.

The regions bounded by the line segments in each plane are called the faces of the solid.

Solids may have flat surfaces or curved surfaces. This unit contains a discussion of (1) the common solids which are composed of nothing but flat surfaces, (2) some that consist of both flat and curved surfaces, and (3) one which is only a curved surface.

Concept: If a solid consists of nothing but flat surfaces, and if all these surfaces are polygons, then the solid is called a polyhedron.

The simplest polyhedron is the one described above, with four vertices and four triangular faces.

Polyhedrons are classified according to the number of their faces. The four-faced polyhedron is called a tetrahedron. Below are the names of some of the more common polyhedrons.

five faces	pentahedron	nine faces	nonahedron
six faces	hexahedron	ten faces	decahedron
seven faces	heptahedron	twelve faces	dodecahedron
eight faces	octahedron	twenty faces	icosahedron

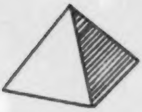
If the pupils mastered the names of the common polygons last year, they should have no difficulty mastering the names of these common polyhedrons. Except for the quadrilateral, the prefixes of the names of the common polygons are the same as the prefixes for these common polyhedrons.

Certain polyhedrons are called regular polyhedrons. A regular polyhedron is a polyhedron all of whose faces are regular congruent polygons and whose polyhedral angles are all congruent. A polyhedral angle is the configuration formed by faces of a polyhedron which have a common vertex.

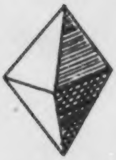
Any polyhedron which is not a regular polyhedron is called an irregular polyhedron.

Concept: There are only five regular polyhedrons: (1) the regular hexahedron (a cube), (2) the regular tetrahedron (a pyramid), (3) the regular octahedron, (4) the regular dodecahedron, and (5) the regular icosahedron.

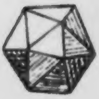
The faces of the regular tetrahedron are triangles. The faces of the regular hexahedron are squares. The faces of the regular octahedron are triangles. The faces of the regular dodecahedron are pentagons. The faces of the regular icosahedron are triangles.



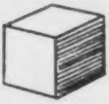
Tetrahedron



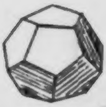
Octahedron



Icosahedron



Hexahedron



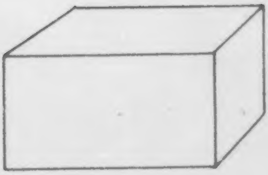
Dodecahedron

Prisms

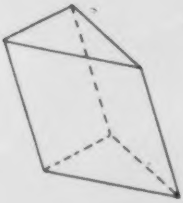
The purpose of this unit is to introduce the concept of prism, right prism, oblique prism, and give the pupil experience at recognizing polyhedrons which are prisms. As a preparation for the introduction of this material, the concepts of parallel, parallelogram, corresponding parts of congruent figures, perpendicular, and the descriptive terms triangular, rectangular, pentagonal, hexagonal, and the like should first be reviewed.

Concept: A prism is a certain type of polyhedron. A prism is a polyhedron with two congruent and parallel faces, similarly placed, called bases, and whose other faces, called lateral faces, are parallelograms formed by joining corresponding vertices of the bases.

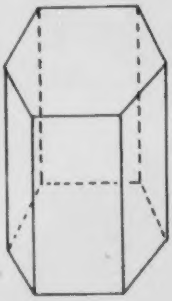
The first three diagrams below represent prisms. The solids represented by the remaining diagrams are not prisms. To the right of each of these last three diagrams is an explanation as to why each cannot be considered to be a prism.



Prism

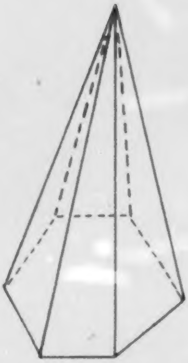


Prism



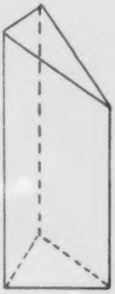
Prism

The solid at the right does not have two parallel bases so it is not a prism.



The solid at the right is not a polyhedron so it is not a prism.

The solid at the right does not have two parallel bases so it is not a prism.



A prism is triangular, rectangular, hexagonal, and so on according to the shape of the bases. Most pupils are familiar with triangular prisms because this type of prism is used a great deal in science classroom demonstrations to refract sunlight and disperse it into its spectrum colors. It should be emphasized that this is only one of many types of prisms.

Prisms may be classified as being right prisms and oblique prisms. If the lateral faces are perpendicular to the bases, the prism is a right prism. If the lateral faces are not perpendicular to the bases, the prism is an oblique prism.

It would be well to have the pupils experience recognizing right and oblique prisms when viewed from various angles.

Pyramids

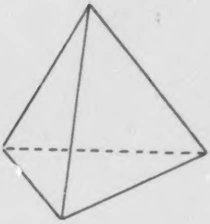
The purpose of this section is to introduce the concepts of pyramid, regular and irregular pyramid, the difference between a prism and a pyramid, and to give the pupil experience at recognizing pyramids.

Concept: A pyramid is a certain type of polyhedron. A pyramid is a polyhedron with one face a polygon and the other faces triangles with a common vertex.

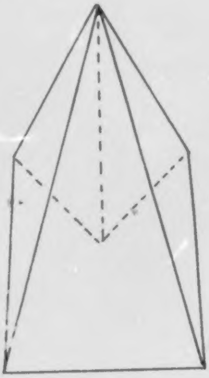
A pyramid is a polyhedron and a prism is a polyhedron, but a pyramid cannot be a prism and a prism cannot be a pyramid. A prism has two congruent and parallel bases. If two faces are parallel, the remaining sides cannot be triangles with a common vertex.

In the pyramid the polygon is called the base and the triangles are called the lateral faces. The common vertex of the lateral faces is called the vertex or apex of the pyramid.

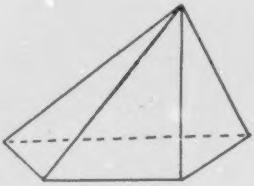
Each of the figures below, except the last one, represents a pyramid. Each of these pyramids has one face a polygon and the remaining faces triangles with a common vertex.



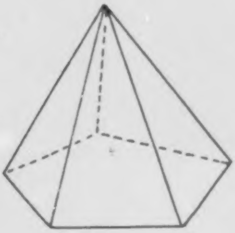
Pyramid



Pyramid

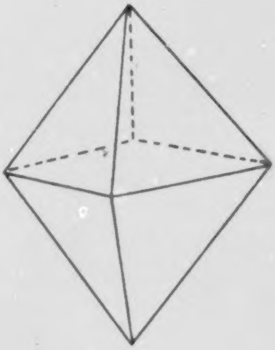


Pyramid



Pyramid

The diagram at the right does not represent a pyramid. If one face is selected as the base, the remaining triangular faces do not have a common vertex.

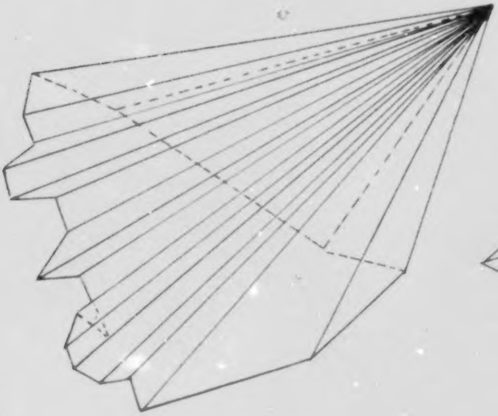


The last figure cannot be classified as a pyramid, because, although any face chosen as the base is a polygon and the remaining faces are triangles, these remaining triangular faces do not have a common vertex.

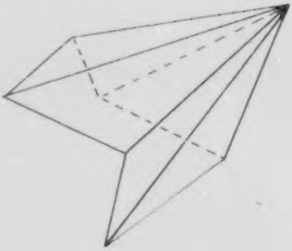
The altitude of a pyramid is the segment from the vertex of the pyramid perpendicular to the plane of the base. The altitude can also be defined as the length of such a segment or the perpendicular distance from the vertex to the plane of the base. Pyramids may be classified as regular pyramids or irregular pyramids.

If the base of the pyramid is a regular polygon and if the altitude passes through the center of the base, then the pyramid is a regular pyramid. If the base is not a regular polygon or if the altitude does not pass through the center of the base, then the pyramid is an irregular pyramid. In the preceding diagrams, the two pyramids at the left represent regular pyramids. The remaining two pyramids illustrate irregular pyramids.

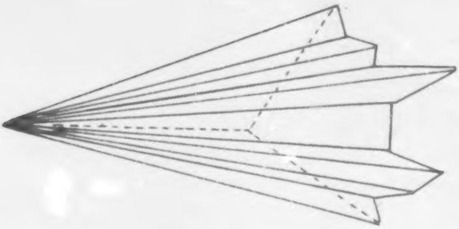
There is no restriction on the shape of the base of a pyramid except that it must be a polygon. Also, there is no restriction as to the location of the vertex of the pyramid as long as it is not located on the plane of the base. It is therefore possible to have rather odd-shaped solids that are pyramids. The figures below and on the next page are a few examples.



Irregular pyramid



Irregular pyramid



Irregular pyramid

Cones

The cone is the first solid studied in this unit which is not a polyhedron. A cone has one flat surface and one curved surface.

Concept: Consider a closed curved line in a plane and a point not on that plane. Imagine all the possible line segments drawn to every point on the closed curve from the given point. The surface formed by these line segments plus the region bounded by the given closed curve is called a cone.

The region bounded by the closed curve is called the base of the cone and the curved surface is called the lateral surface. The given point which was not on the plane of the base is called the vertex or apex of the cone.

A cone is most often thought of as having a circular base. This is not a requirement of a cone. The base of a cone can have any shape as long as it is a region bounded by a closed curved line in a plane. If the base of a cone is circular, the cone is called a circular cone.

The altitude of a cone is the segment from the vertex perpendicular to the plane of the base. The altitude can also be defined as the length of such a segment or it can be defined as the perpendicular distance from the vertex to the base.

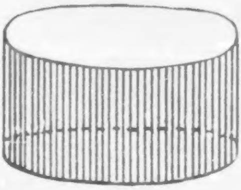
Cylinders

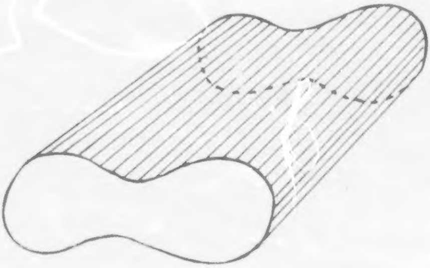
The purpose of this section is to introduce the concepts of cylinder, circular cylinder, and right and oblique cylinder.

Concept: A cylinder is the solid formed by the union of the regions enclosed by two congruent closed curved lines contained in *parallel planes and similarly placed and the surface formed by all the line segments joining corresponding points on the two congruent curved lines.*

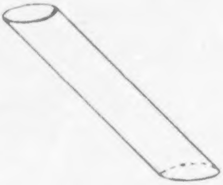
A cylinder is not a polyhedron. It has two flat surfaces and one curved surface. The comparison of a cone and a cylinder is much the same as the comparison of a pyramid and a prism. The comparison of the relationship of the cone to the cylinder and the relationship of the pyramid to the prism should be made evident to the pupils at this point because such comparison is made again later in the section on volumes.

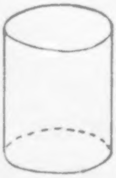
A cylinder is most often thought of as a solid with circular bases. However, there is no requirement that the bases of a cylinder be circular. Like a cone, the bases of a cylinder may be of any shape as long as they are regions bounded by closed curved lines in parallel planes, are congruent and are similarly placed. Below are two examples of cylinders whose bases are not circular.





A circular cylinder is a cylinder which has circular bases. A cylinder whose lateral surface is perpendicular to the bases is called a right cylinder. If the lateral surface is not perpendicular to the bases, the cylinder is an oblique cylinder. Below is an example of an oblique circular cylinder and a right circular cylinder.





Spheres

The concepts contained in this section are those of a sphere, interior of a sphere, exterior of a sphere, radius of a sphere, diameter of a sphere, hemisphere, great circles, and small circles.

A circle was defined as the set of all points in a plane a given distance from a given point.

Concept: A sphere is the set of all points a given distance from a given point called the center of the sphere.

A sphere is a surface. It is a closed surface completely enclosing a portion of space, and is therefore a solid.

Concept: The radius of the sphere is the distance from the center to any point on the sphere.

Radius is also the name given to any line segment whose endpoints are the center and a point on the sphere, and it is the name given to the length of such a segment.

The interior of the sphere is the set of all points whose distance from the center is less than the radius.

The exterior of the sphere is the set of all points whose distance from the center is greater than the radius.

Concept: A diameter of a sphere is any segment whose endpoints are points on the sphere and which contains the center of the sphere.

It is also the length of such a line segment or the distance between the endpoints of such a line segment.

A hemisphere is half a sphere bounded by a great circle.

Great circles and small circles are certain parts of spheres. A good method of introducing the topic of great circles is by first discussing the intersection of a plane and a sphere. If a plane intersects a sphere, the intersection consists of either a single point or of a circle. If the intersection consists of a circle and if the plane contains the center of the sphere, then the circle is called a great circle. A great circle is the set of all points contained in the intersection of a sphere and a plane containing the center of the sphere.

If the intersection of a sphere and a plane consists of a circle, and if that plane does not contain the center of the sphere, then the circle forming the intersection is called a small circle.

If the surface of the earth were a sphere, then the earth's equator would be a great circle. Its plane would contain the center of the sphere.

Surface Areas of Solids

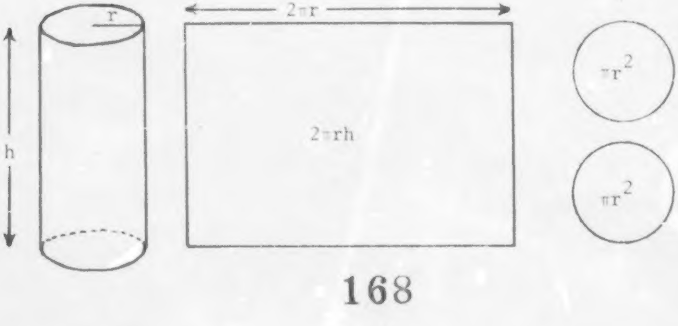
Concept: The surface area of a polyhedron is the sum of the areas of all its faces.

Students should be introduced to this concept, but at this level, only the surface area of a *rectangular prism* need be considered.

As enrichment material, the surface area of a right circular cylinder may be developed. The bases of a right circular cylinder are circular regions whose area can be calculated by the equation $A = \pi r^2$. To demonstrate one way of calculating the area of the lateral surface of the cylinder, cut the lateral surface of a right circular paper cylinder perpendicularly from one base to the other base and lay it out flat. This flattened lateral surface will form a rectangle.

The area of a rectangle is the product of the length and the width. In the case of this rectangle, its length is equal to the circumference of the base of the cylinder and its width is equal to the height of the cylinder. The area of the rectangular region is therefore equal to $2\pi rh$, where r is the radius of the circular base and h is the height of the cylinder.

The area of each base of the cylinder is equal to πr^2, so the entire area of the cylinder is equal to the sum of the areas of the two bases ($2\pi r^2$) and the area of the lateral surface ($2\pi rh$). Thus $A = 2\pi r(r + h)$, where r is the radius of the circular base and h is the height of the cylinder, or the perpendicular distance from the plane of one base to the plane of the other base.



Volumes of Solids

(In the eighth grade, or as part of a second cycle devoted to geometry in the seventh grade, the topic, volumes of solids, may be extended beyond the volume of a rectanglar prism.)

The concepts introduced in this section include volume, volume of a prism, volume of a cylinder, volume of a cone and volume of a pyramid. The material to be reviewed as a preparation for this new work should include basic units of volume and the volume of a rectangular prism.

A geometric solid is a surface. Therefore, when we speak of the volume of a solid, we are actually referring to the volume of the space enclosed by a solid. In this section, when reference is made to the volume of any solid, it is understood that reference is actually being made to the volume of the interior or the space bounded by that solid.

The topic of volumes should begin with a reveiw of the basic units of volume, including the cubic inch, cubic foot, and cubic centimeter. Models of these units are useful. These and other models may be used to give a concept of the relative sizes of the various units used for expressing volumes.

After the basic units of measurement of volumes have been reviewed, the topic of the volume of a rectangular prism should be reviewed. Again, physical models are helpful.

The equations to be used in calculating the volumes of most of the solids can be developed only intuitively at this grade level. This includes the equation for the volume of any prism.

In presenting the concept of the volume of a prism, first consider a prism whose altitude (perpendicular distance between the parallel planes containing the bases) is one inch. If the area of the base is X square inches, then the number of cubic inches which could be made to fit on this base would be X number of cubic inches. If the altitude is doubled, the number of cubic inches contained in the volume will be $2X$. The pupil should soon be able to see that the volume of the prism will be equal to the area of the base multiplied by the altitude. This can be expressed by the equation $V = Bh$, where B represents the area of the base and h the altitude of the prism.

This equation is valid for all prisms, both right and oblique.

The equation to be used in calculating the volume of a circular cylinder can be developed in the same way as that for the volume of the prism. The result is the same equation $V = Bh$. In the case of the circular cylinder, the area of the base is calculated by use of the equation for the area of a circular region $A = \pi r^2$. Therefore, in the case of the circular cylinder, the equation for the volume may be written either as $V = Bh$ or $V = \pi r^2 h$, where r is the radius of the circular base and h is the altitude.

Experimentation can be used to develop the concept of the equation to be used in calculating the volumes of pyramids and cones. This involves the selection of an open pyramid and an open prism whose bases have the same area and whose altitudes are equal. Fill the pyramid with water or sand and pour into the prism. Repeat this until the prism is full. It should take exactly three pyramids of water or sand to fill the prism. This demonstrates the fact that the volume of a pyramid is equal to one-third the volume of a prism which has a base the same area as the pyramid and whose altitude is equal to that of the pyramid.

The equation for the volume of a pyramid is therefore $V = \frac{1}{3}Bh$.

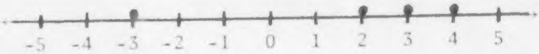
The procedure can then be repeated using a cone and a cylinder whose base and whose altitude are each equal to that of the cone. It should take exactly three cones of water or sand to fill the cylinder, demonstrating that the volume of the cone is equal to one-third that of the cylinder. The equation for the volume of the cone is equal to one-third that of the cylinder. The equation for the volume of the cone is therefore $V = \frac{1}{3}Bh$ or $V = \frac{1}{3}\pi r^2 h$.

Space Visualization

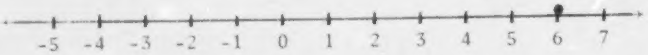
A unit on solid geometry may include a section of space visualization, pertaining to prisms, cylinders, cones and pyramids. Students may be given exercises in drawing representations of solids; making models, for which patterns are available in reference materials; and drawing representations of the intersection of a plane and various solids. Some of the better students may be challenged by exercises to develop their own patterns for making solid models.

Coordinate Geometry Using the Number Line

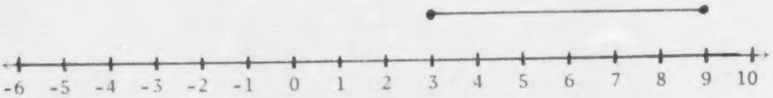
Review the number line for integers and rationals, stressing the equality of divisions for the integral points. Recall that sets of points may be graphed. The set {-3,2,3,4} is graphed



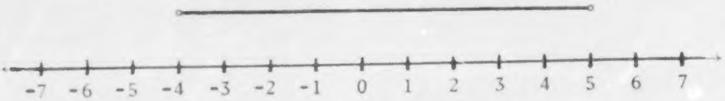
The student should be familiar with set builder notation. Thus, $\{a|a = 6\}$ is read as the set of all a such that a = 6 and is graphed as



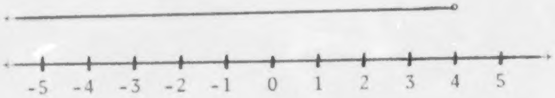
Also, $\{x|3 \leq x \leq 9\}$ is read as the set of all x such that x is greater than or equal to 3 *and* less than or equal to 9. The graph is



Solid large dots at the ends of the line segments indicate that the end points are included in the graphed set. Again, $\{y|-4<y<5\}$ is read as the set of all y such that y is greater than -4 and less than 5. The graph is



The circles at the ends of line segments indicate that the end points are *not* included in the graphed set. Finally, $\{y|y<4\}$ is read as the set of all y such that y is less than 4 and graphed as



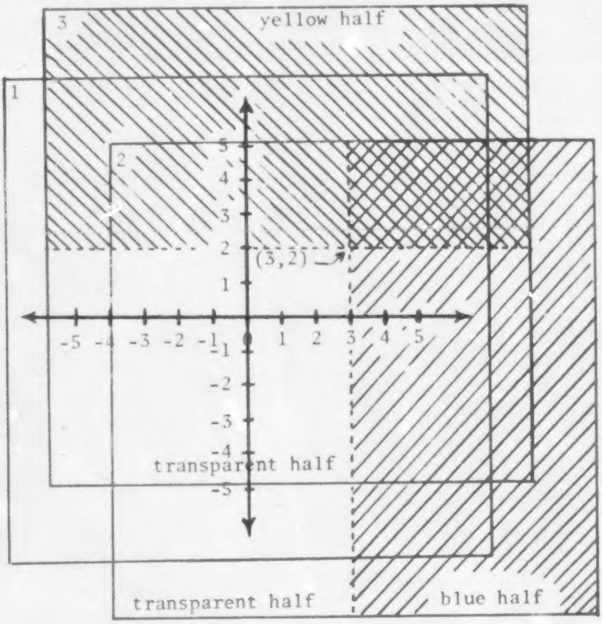
Coordinate Geometry Using the Number Plane

Some practice with the common number-letter grid on a road map, used to locate the region in which a required city or town is found, may be used to introduce the concept that *two* items are needed to locate a position on a plane. Points on a number plane are identified by using two number lines intersecting at right angles. The same units of division should be used on each number line. By convention the x axis is drawn left to right and the y axis is drawn perpendicular to x axis. Arrows must be drawn at each end of axes. The point where x and y axes intersect is called the origin and is the zero point on each number line.

Concept: Each point in the number plane is identified by an ordered pair of real numbers with the x value given first, then the y value: (x,y), called the coordinates of the point.

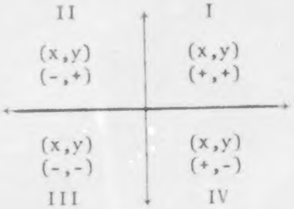
There are several ways of visualizing the manner in which the coordinates are used to locate the position of a point on the coordinate plane.

One useful way is to consider the point as being located by the intersection of two perpendiculars. One perpendicular is erected at a point on the x axis given by the x coordinate, and the other perpendicular is erected at a point on the y axis given by the y coordinate. The figure below shows how the overhead projector and three transparencies may be used to give practice in locating a point on the number plane. The first transparency has the two perpendicular axes drawn on it. The second is a sheet which is divided "vertically", half transparent and half blue; and the third is a sheet which is divided "horizontally", half transparent and half yellow. The first transparency is kept fixed and the other two are moved over it to locate points in the number plane.



The above diagram shows the transparencies arranged to locate the point (3,2) on the number plane.

The axes divide the plane into four quadrants, which may be identified according to signs of the numbers in the ordered pairs. All values of x to the right of the y axis are positive and to the left negative. All values of y above the x axis are positive and below negative.



Identify the quadrant in which each of the following ordered pairs lies:

(a) (4,5), (b) (-3,2),

(c) (-5,-3), (d) (7,-5)

Answers:

(a) I, (b) II, (c) III, (d) IV

Points may be chosen so that when straight lines are drawn rectangles, and other geometric figures may be identified.

Plot on a sheet of graph paper each of the following sets of points which form the vertices of polygons. In each case, join the points in the order given, join the last point to the first, identify the polygon formed.

(a) (0, 0), (+4, 0), (+4, +3)
(b) (-5, -2), (+5, -2), (+6, +3), (-6, +3)
(c) (-1, 0), (+4, 0), (+5, +4), (0, +4)
(d) (-4, -5), (-1, -5), (-1, -2), (-4, -2)
(e) (0, 0), (7, 0), (7, 4), (0, 4)

Answers: (a) Right triangle (b) Trapezoid (c) Parallelogram
(d) Square (e) Rectangle

Perimeters and areas of rectangles and squares may be found using coordinates, taking care that vertices chosen will make the sides parallel to the axes.

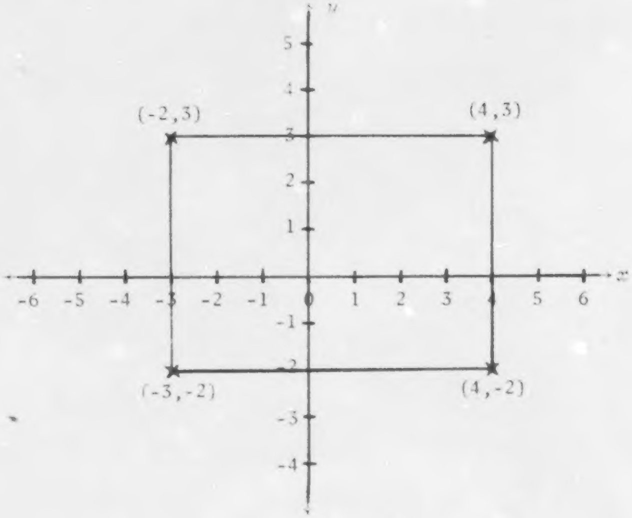
Plot the points, (-3,-2), (4,-2), (4,3), (-3,3), and join the points in order.

To find the perimeter and the area of a rectangle it is necessary to have the measure of its length and width.

Counting the units in the diagram on the next page we have $l = 7$ and $w = 5$.

Then $P = 2(l+w)$
$P = 2(7+5)$
$P = 2(12) = 24$ units.

$A = lw$
$A = 7(5)$
$A = 35$ square units

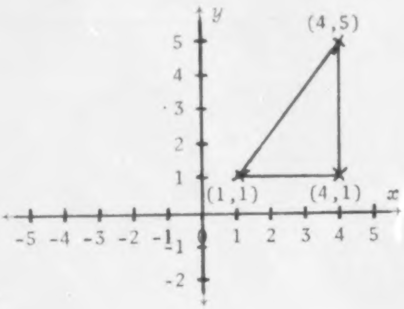


Right triangles may be graphed - again with judicious choosing of vertices in order that the two legs (base and altitude) are parallel with the axes. (The altitude to the hypotenuse is not considered in problems at this level of coordinate geometry.) Perimeters may be found, using the Pythagorean theorem to find the hypotenuse, using only values which do not result in irrationals. [Three of the many Pythagorean triples are (3,4,5) (5,12,13) (7,24,25). Any multiple of the sets will result in rational values for the three sides.]

Plot the points (1,1), (4,1), (4,5) and joint the points in order.

In order to find the perimeter of the triangle the measures of the lengths of the three sides are necessary.

$a^2 + b^2 = c^2$
$3^2 + 4^2 = c^2$
$9 + 16 = c^2$



$$25 = c^2$$

$$5 = c$$

Then $P = 3+4+5 = 12$ units

In order to find the area, the measures of the base and altitude are necessary. We have $b = 3$ units, $h = 4$ units.

$$A = \frac{1}{2}bh$$

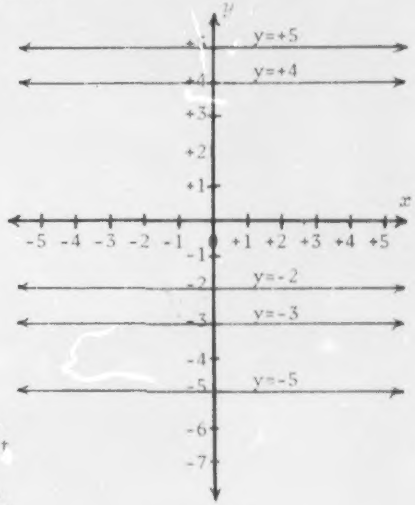
$$A = \frac{1}{2}(3)(4)$$

$$A = \frac{12}{2} = 6 \text{ square units}$$

Graphing Open Sentences

In the graphing of open sentences, the first graphs may be those which result in lines parallel to the x-axis or y-axis.

Concept: The x-axis is the line containing the set of all points such that $y = 0$; the y-axis is the line containing the set of all points such that $x = 0$.



Using a set of coordinate axes, draw and label the following sets of points.

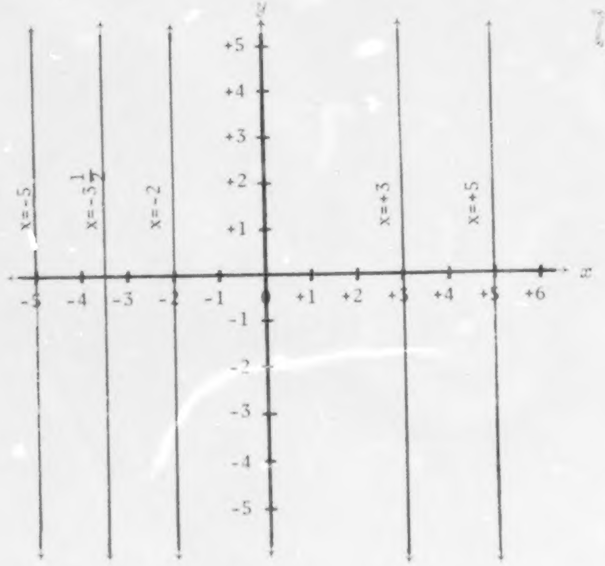
(a) The set of all points 4 units above the x-axis
(b) The set of all points 2 units below the x-axis
(c) The set indicated by the equation $y = +5$
(d) The set indicated by the equation $y = -3$
(e) The set indicated by the equation $y = -5$

The arrows at the ends of the lines which represent the several sets indicate that there is neither a greatest nor a least x coordinate of the points in these sets. Thus some points in the coordinate plane which *satisfy* $y = -3$, for instance, are (-50,000, -3), (-10, -3), (-5, -3), (8, -3), (25, -3) and (1,000,000, -3). Contrast this with graphing open sentences on the *number line*.

Using a set of coordinate axes, draw and label the following sets of points.

(a) The set of all points 3 units to the right of the y-axis
(b) The set of all points 2 units to the left of the y-axis
(c) The set indicated by the equation $x = +5$
(d) The set indicated by the equation $x = -5$
(e) The set indicated by the equation $x = -3\frac{1}{2}$

Answer:

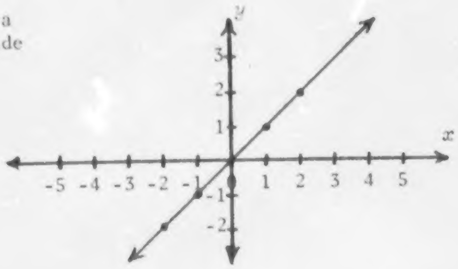


Graphing - Equalities

In graphing equalities a table of values should be made first.

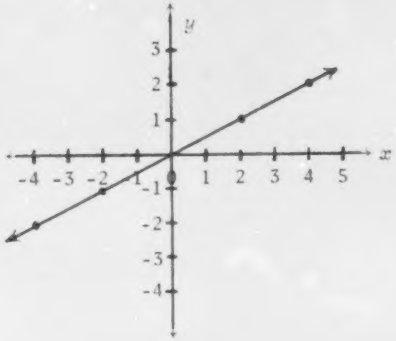
Graph x = y

x	y
-2	-2
-1	-1
0	0
1	1
2	2



Graph $x = 2y$

y	x
-2	-4
-1	-2
1	2
2	4

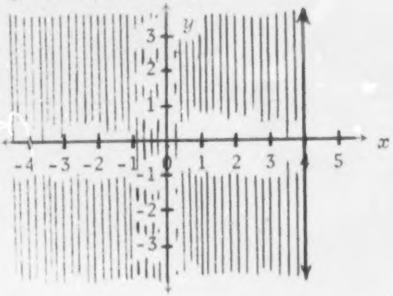


Graphing - Inequalities

Concept: The graph of an inequality consists of part of the plane bounded by one or two lines.

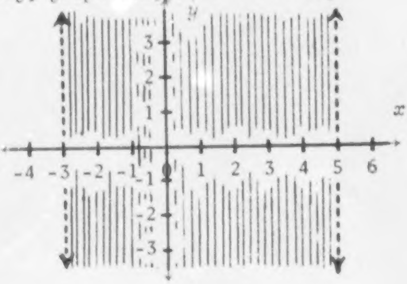
Graph $x \leq 4$. [*More precisely, graph* $\{(x,y)|x \leq 4\}$]

Answer: This consists of all points to the left of the y-axis, the y-axis, and up to and including four units to the right of y-axis. The solid line at x = 4, indicates by convention that x = 4 is included in the graph of x ≤ 4.

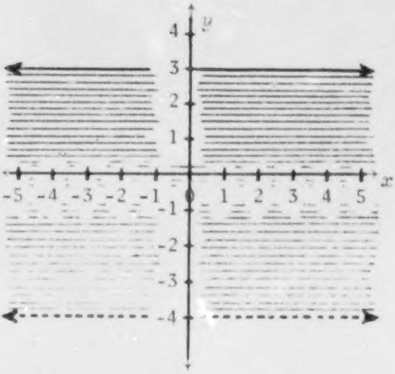


Graph $-3 < x < 5$. [*More precisely, graph* $\{(x,y)|-3 < x < 5\}$]

Answer: This consists of all points up to but not including -3 to the left of the y-axis, the y-axis, and all points up to but not including 5 to the right of the axis. The dashed boundary: x = -3 and x = +5, indicates, by convention, that x = -3 and x = +5 are not included.



Graph $-4 < y \leq 3$

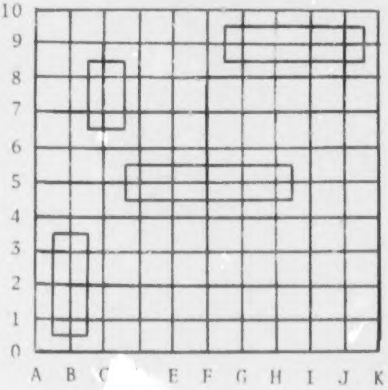


Answer: This consists of all points below the x-axis up to but not including $y = -4$, the x-axis, and all points above the x-axis up to *and* including $y = 3$. The dashed line at $y = -4$ indicates that $y = -4$ is not included, while the solid line at $y = 3$ means that $y = 3$ is included in the graph.

There are a number of enrichment exercises which involve the use of graphs. Some of these follow.

I. Using lined graph paper, follow the directions below for playing the Game of Battleship or Sinking Ships.

This game, best played by two pupils or two teams of pupils, emphasizes the location of a place by coordinates. On one chart each pupil outlines his four ships as shown, making sure that his opponent cannot see this chart. Ships may be placed in a horizontal or vertical position, and can be referred to in order of size: carrier (5 consecutive intersections), battleship (4 consecutive intersections), destroyer (3 consecutive intersections), submarine (2 consecutive intersections).



A Possible Diagram for the Game of Battleship

One contestant starts with a salvo of four shots in an attempt to locate his opponent's ships. As he announces each shot, such as B5 or H8, his opponent tells whether it is a miss or a hit. If a hit is scored, the opponent tells the type of ship hit. The offensive player then records his shot on his blank chart. For a miss he records an X. For a hit he records a number indicating the number of squares occupied by his opponent's ship. The defensive player records each of his opponent's shots by writing an X on the chart on which his own ships are marked.

After one contestant completes the first salvo, the opponent takes his turn with four shots. Play continues by turns until one player has sunk all his opponent's ships. A ship is sunk only when the number of hits is the same as the number of intersections it occupies. However, if it is the person or team who began the game who first succeeds in sinking all his opponent's ships, the second person is entitled to an equal number of shots in an attempt to tie the score or, in case he needs fewer shots than did his opponent, to win.

Duplicate copies of the blank chart might be furnished by the teacher.

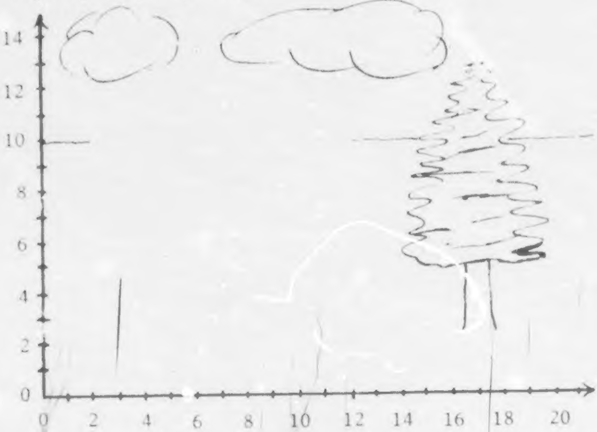
II. There are available many outlined figures containing appended sets of coordinate points which when graphed will produce a picture. Students enjoy this type graphing and while having fun they are also learning.

For example, in the following exercise, if the points are connected as indicated a house is added to the scene.

Locate the following points on the diagram below and connect each point, in each group, to the succeeding point in the order listed.

(a) (1,8) (3,12), (5,7), (13,8), (13,3), (5,1), (5,7), (1,8) (1,3), (5,1)

(b) (3,12), (11,12), (13,8)



Glossary of Terms for Geometry

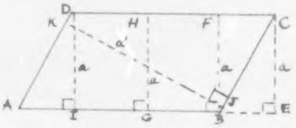
Altitude of a cone. The segment from the vertex perpendicular to and terminated by the plane of the base or the perpendicular distance from the vertex to the plane of the base.

Altitude of a cylinder. A segment from the plane of one base perpendicular to and terminated by the plane of the other base or the perpendicular distance between the planes of the bases.

Altitude of a parallelogram. A perpendicular line segment joining a point on any side of a parallelogram with the line containing the opposite side.

The length of this line segment.

The line containing this line segment.



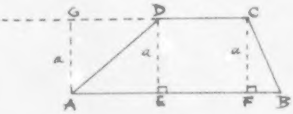
Altitude of a prism. A segment from one base perpendicular to and terminated by the plane of the other base, or the perpendicular distance between the planes of the bases.

Altitude of a pyramid. The segment from the vertex perpendicular to and terminated by the plane of the base or the perpendicular distance between the vertex and the plane of the base.

Altitude of a trapezoid. A perpendicular line segment joining a point on one of the parallel sides with the line containing the opposite side.

The length of this line segment.

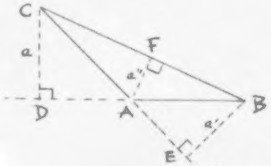
The line containing this line segment.



Altitude of a triangle. The perpendicular line segment joining any vertex with the line containing the opposite side.

The length of this line segment.

The line containing this line segment.



Angle. The set of all points contained on two rays which have the same endpoint.

Angles (adjacent). Two angles in the same plane having a common ray and a common vertex and whose interiors have no point in common.

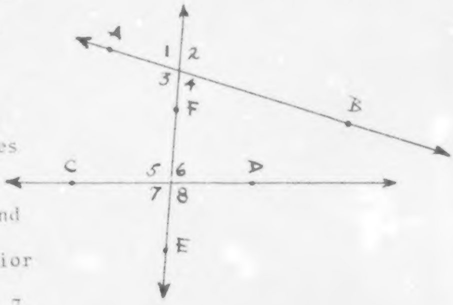
Angle (acute). Any angle whose measure is less than that of a right angle.

Angles (complementary). Two angles such that the sum of their measures is equal to 90°. The two acute angles of a right triangle are always complementary.

Angles (equal). Angles having the same measure.

Angle (exterior). An angle such that one side is the ray containing one side of a polygon and the other side is the ray which is opposite to that which contains an adjacent side of the polygon.

Angles (made by a transversal).



In the figure, lines AB and CD are intersected by the transversal EF. Angles 3, 4, 5, and 6 are interior angles. Angles 1, 2, 7, and 8 are exterior angles. Angles 3 and 6, and angles 4 and 5 are the pairs of alternate interior angles. Angles 1 and 8, and angles 2 and 7 are the pairs of alternate exterior angles. Angles 1 and 5, angles 2 and 6, angles 3 and 7, and angles 4 and 8 are corresponding angles.

Angle (measure of). The measure of an angle is the measure of the amount of rotation of the initial side of the angle about its vertex necessary to make the initial side coincide with the position of the terminal side. The unit of measurement is the degree which is $\frac{1}{360}$ of a complete rotation. In more advanced work, counterclockwise rotation is taken as positive.

Angle (obtuse). Any angle whose measure is greater than that of a right angle and less than that of a straight angle.

Angle (reflex). Any angle whose measure is greater than that of a straight angle but less than 360°.

Angle (right). Any angle whose measure is one-half that of a straight angle, or 90°.

Angle (straight). Any angle whose sides lie on the same straight line extending in opposite directions from the vertex. Any angle whose sides are opposite rays. Any angle whose measure is 180°.

Angles (supplementary). Any two angles such that the sum of their measures is equal to 180°.

Angles (vertical). Two angles such that the sides of one are rays opposite to the sides of the other.

Arc. A segment of a curve, particularly a circle. The length of such a segment.

Area. The number of times a given unit (such as a square inch) is contained in the given region.

Areas of plane geometric figures.

Circle $A = \pi r^2$
Parallelogram $A = bh$
Rectangle $A = lw$
Trapezoid $A = \frac{1}{2} h\ (b + b')$
Triangle $A = \frac{1}{2} bh$

Areas of solid geometric figures.

Cylinder $A = 2\pi r^2 + 2\pi rh$ or $A = 2\pi r(r+h)$

Cone $A = \pi r^2 + \pi rs$ or $A = \pi r(r + s)$

Axiom. An assumption.

Axis (coordinate). A line along which or parallel to which a coordinate is measured. The horizontal axis is usually referred to as the x-axis and the vertical axis as the y-axis.

Axis of a right circular cylinder. The segment joining centers of the bases, or the line containing such a segment.

Axis of a right circular cone. The segment joining the vertex to the center of the base, or the line containing such a segment.

Bisect an angle. To determine a ray in the interior of the angle which forms two angles of equal measure with the side of the original angle.

Bisect a line segment. To determine a point on the line segment equally distant from the endpoints.

Centimeter. One hundredth part of a meter.

Chord of a circle. Any straight line segment whose endpoints are points on the circle.

Circle. The set of all points in a plane a fixed distance, called the radius, from a fixed point, called the center.

Circumference of a circle. The length of a circle.

Closed circular region. The set of all points on a circle and in the interior of the circle. The region bounded by a circle including the circle itself.

Closed plane curve. A curve which completely bounds a finite portion of a plane called a region.

Collinear points. Points lying on the same line.

Compass (more correctly a pair of compasses). An instrument used for measuring distances between two points or for describing circles.

Concentric circles. Circles lying in the same plane having a common center.

Concurrent lines. Lines which have a point in common.

Concurrent planes. Three or more planes which have a point in common.

Cone. A solid consisting of the union of the region enclosed by a closed curved line in a plane and the surface formed by segments from every point on that line to one point not on the plane of that line.

Cone, circular. A cone with a circular base.

Cone, oblique. A cone whose axis is not perpendicular to the base.

Cone, right. A cone whose axis is perpendicular to the base.

Congruent figures. Figures which can be superposed, that is, placed one upon the other so that corresponding parts coincide. Sometimes taken as an undefined concept.

Coplanar lines. Lines in the same plane.

Coplanar points. Points in the same plane.

Cube. A solid bounded by six planes, with its twelve edges all equal and its face angles all right angles.

Cubic centimeter. The volume of a cube each of whose edges is one centimeter in length.

Cubic inch. The volume of a cube each of whose edges is one inch in length.

Curve. A curve is a path consisting of a set of points. In higher mathematics a curve includes a straight line, but in this unit a curve will be described as a path no part of which is a straight line segemnt.

Cylinder. The solid formed by the union of the regions bounded by two congruent closed curves in parallel planes and similarly placed, and the surface formed by the parallel segments joining all corresponding points on the congruent closed curves.

Cylinder (circular). A cylinder with circular bases.

Cylinder (oblique). A cylinder whose axis is not perpendicular to the bases.

Cylinder (right). A cylinder whose axis is perpendicular to the bases.

Cylinder (right circular). The surface generated by revolving a rectangle about one of its sides. The solid bounded by such a surface.

Decagon. A polygon having ten interior angles. A polygon having ten sides.

Decahedron. A polyhedron with ten faces.

Degree. A unit of angular measure. See Angle, measure of.

Diagonal of a polygon. Straight line segment connecting two non-adjacent vertices.

Diameter of a circle. Any chord containing as one of its points the center of the circle.

Diameter of a sphere. A segment whose endpoints are on the sphere and which contains the center of the sphere.

Dodecagon. A polygon having twelve interior angles. A polygon having twelve sides.

Dodecahedron. A polyhedron with twelve faces.

Dodecahedron, regular. A dodecahedron all of whose faces are congruent regular pentagons and all of whose polyhedral angles are congruent.

Edge. A line segment which is the intersection of two plane faces of a solid.

Exterior of a circle. The set of all points in the plane of the circle whose distance from the center is greater than the radius of the circle.

Exterior of a sphere. The set of all points in space whose distance from the center is greater than the radius of the sphere.

Face of a polyhedron. The region enclosed by a bounding polygon of a polyhedron.

Foot. A unit of linear measure equal to 12 inches. The point of intersection of a line with another line or a plane.

Great circle. The circle formed by the intersection of a sphere and a plane which contains as one of its points the center of the sphere.

Hemisphere. A half of a sphere bounded by a great circle.

Hepta-. Prefix meaning seven, as in *heptagon* (polygon having seven sides) and *heptahedron* (polygon having seven faces).

Hexagon. A polygon having six interior angles. A polygon having six sides.

Hexahedron. A polyhedron of six faces.

Hexahedron (regular). A hexahedron all of whose faces are squares and all of whose polyhedral angles are congruent.

Horizontal. In a plane perpendicular to a plumb line.

Hypotenuse. The side opposite the right angle in a right triangle.

Icosa-. Prefix meaning twenty, as in *icosagon* (polygon having twenty sides) and *icosahedron* (polyhedron having twenty faces).

Icosahedron (regular). An icosahedron all of whose faces are equilateral triangles and all of whose polyhedral angles are congruent.

Inch. A unit of measure of distance or length equal to one-twelfth of a foot or $\frac{1}{36}$ of a yard.

Interior of a circle. The set of all points in the plane of the circle whose distance from the center is less than the radius of the circle.

Interior of a sphere. The set of all points in space whose distance from the center is less than the radius of the sphere.

Kilometer. A unit of measure of length or distance equal to one thousand meters.

Leg of a right triangle. Either one of the sides adjacent to the right angle.

Line. An undefined geometric configuration whose properties are length and position.

Line (broken). A succession of straight line segments connected end to end, not all of which are in the same straight line.

Line (closed broken). Any broken line which forms a closed path.

Lines (coinciding). Lines having identical sets of points.

Lines (concurrent). See concurrent.

Lines (intersecting). Lines which have one and only one point in common.

Line (horizontal). Any line which is perpendicular to a given vertical line.

Lines (parallel). Lines in the same plane which do not intersect.

Line segment. A finite portion of a line containing the points between two given points, and including the two given points.

Lines (skew). See skew lines.

Line (vertical). The plumb line.

Line (plumb). The line in which a string hangs when supporting a weight at one end.

Median of a triangle. The line segment joining a vertex to the middle point of the opposite side.

Meter. The basic unit of linear measure of the metric system; the distance between two marks on a platinum bar preserved in Paris. It is equal to 39.37+ inches.

Millimeter. One thousandth part of meter.

Nona-. Prefix meaning nine, as in *nonagon* (polygon having nine sides) and *nonahedron* (polyhedron having nine faces).

Oblique. Neither perpendicular nor parallel, such as oblique lines and oblique planes.

Octa-. Prefix meaning eight, as in *octagon* (polygon having eight sides) and *octahedron* (polyhedron having eight faces).

Octahedron (regular). An octahedron all of whose faces are congruent equilateral triangles and all of whose polyhedral angles are congruent.

Parallel lines. See lines, parallel.

Parallel planes. See planes, parallel.

Parallelogram. A quadrilateral with its opposite sides parallel. The opposite sides lie on parallel lines.

Penta-. Prefix meaning five, as in *pentagon* (polygon having five sides) and *pentahedron* (polyhedron having five faces).

Perpendicular. Two straight lines are perpendicular if they form a pair of equal adjacent angles. The property of lines and planes which intersect at an angle of 90 degrees.

Perimeter. The length of a closed curve or a closed broken line. The sum of the lengths of the sides of a polygon.

Pi. The name of the Greek letter π which corresponds to the Roman P. The symbol π denotes the ratio of the circumference of a circle to its diameter. It is an *irrational number* whose common rational approximations include $\frac{22}{7}$, 3.14, 3.1416, and 3.14159.

Plane. An undefined unbounded geometric configuration which has the properties of length and width but has no thickness.

Plane figure. A geometric figure every point on which is contained in the same plane.

Plane (horizontal). A plane perpendicular to a plumb line.

Planes (parallel). Two planes which do not intersect.

Plane (vertical). A plane containing as one of its lines a vertical line.

Point. An undefined geometric configuration which is nondimensional. It has the property of position.

Points (collinear). See collinear.

Polygon. A plane geometric figure formed by a closed broken line.

Polygon (equiangular). A polygon whose interior angles have the same measure.

Polygon (equilateral). A polygon whose sides are equal in length.

Polygon (regular). A polygon whose sides are equal in length and whose interior angles are equal in measure. A polygon which is both equiangular and equilateral.

Polyhedral angle. The configuration formed by the lateral faces of a polyhedron which have a common vertex.

Polyhedron. A solid which is the union of regions, called faces, bounded by polygons.

Polyhedron (regular). A polyhedron all of whose faces are regular polygons and all of whose polyhedral angles are congruent.

Postulate. An assumption.

Prism. A polyhedron with two congruent, parallel and similarly placed faces, called bases, and whose other faces, called lateral faces, are parallelograms formed by joining corresponding vertices of the bases.

Prism (oblique). A prism whose lateral faces are not perpendicular to the bases.

Prism (right). A prism whose lateral faces are perpendicular to the bases.

Protractor. A semicircular plate, or circular clear plastic, graduated in degrees and used to measure angles.

Pyramid. A polyhedron with one face a polygon and the other faces triangles with a common vertex.

Pyramid (regular). A pyramid whose base is a regular polygon and whose altitude passes through the center of the base.

Pythagorean theorem. The sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse.

Quadrilateral. A polygon with four sides.

Radius of a circle. The straight line segment whose endpoints are the center of the circle and a point on the circle; the length of any such line segment.

Radius of a sphere. Any straight line segment whose endpoints are the center of the sphere and a point on the sphere; the length of any such line segment.

Ray. If points A and B are on the same line, the set of all points on the same side of A as B and including the point A form the ray AB.

Rays (opposite). If points A, B, C are on the same line, and A is between B and C then ray AB and ray AC are opposite rays.

Rectangle. A parallelogram with one angle a right angle and therefore all of its angles right angles; a quadrilateral all of whose angles are right angles.

Rhombus. A parallelogram with adjacent sides equal.

Semicircle. One-half of a circle; either of the parts of a circle cut off by a diameter. Not to be confused with a semicircular region.

Similar figures. Figures having all corresponding angles equal and all corresponding line segments proportional.

Skew lines. Nonintersecting, nonparallel lines in space.

Small circle. The intersection of a sphere and a plane which intersects the sphere but which does not contain the center of the sphere.

Solid. A closed surface completely enclosing a finite portion of space.

Space. A three-dimensional unbounded region.

Sphere. The set of all points in space, a fixed distance from a fixed point. Sometimes used as the set of all points whose distance from a fixed point is not greater than a fixed distance.

Square. A rectangle with two adjacent sides equal, therefore all its sides are equal.

Square inch. The area of a square each of whose sides is one inch in length.

Square centimeter. The area of a square each of whose sides is one centimeter in length.

Straight edge. An instrument used for determining points on a straight line identified by two given points.

Surface. A two-dimensional region.

Tetrahedron. A polyhedron of four faces.

Tetrahedron (regular). A tetrahedron all of whose faces are congruent equilateral triangles and all of whose polyhedral angles are congruent.

Transversal. A line intersecting two other lines at two distinct points is a transversal to those lines.

Trapezoid. A quadrilateral which has two and only two sides parallel.

Triangle. A polygon with three interior angles. A polygon with three sides.

Triangle (acute). A triangle each of whose interior angles is acute.

Triangle (equiangular). A triangle all of whose interior angles are equal in measure.

Triangle (equilateral). A triangle all of whose sides are equal in length.

Triangle (isosceles). A triangle with two sides equal in length.

Triangle (obtuse). A triangle one of whose interior angles is an obtuse angle.

Triangle (right). A triangle one of whose angles is a right angle.

Triangle (scalene). A triangle with no two sides equal in length.

Triangular region. A region bounded by a triangle.

Vertex of an angle. The point common to the two rays forming the angle.

Vertex of a polyhedron. The intersection of three or more edges of the polyhedron.

Vertex of a polygon. A point common to two consecutive sides of the polygon.

Vertex angle of a triangle. The angle opposite the base of a triangle.

Vertical line. See line, vertical.

Vertical angles. See angles, vertical.

Volume of a solid. The number of unit cubes which can be contained in the space bounded by the solid.

Volumes of geometric solids.

Circular cone	$V = \frac{1}{3}Bh$ or $V = \frac{1}{3}\pi r^2 h$
Circular cylinder	$V = Bh$ or $V = \pi r^2 h$
Cube	$V = e^3$
Prism	$V = Bh$
Pyramid	$V = \frac{1}{3}Bh$
Rectangular solid	$V = lwh$

XI. STATISTICS

Statistics is a branch of mathematics which deals with the collection, organization, presentation, meaningful analysis, and utilization of data.

The topics in this unit afford opportunities to involve the pupils in interesting, challenging, and enjoyable mathematical activities both inside and outside the classroom. One such activity is the collecting of data, presenting it in the form of graphs, and summarizing it in terms of mean, median, and mode. Such data could be a tabulation of pupils' favorite interests and activities, favorite television programs or stars, or it could be a voter preference in an upcoming school election. This could involve activities in selecting representative samples of a group and performing the sampling.

Statistics can be fun. The pupils may enjoy presenting the same facts in different ways to create different and contradictory impressions. (Some examples of such statements are: "Half of the class scored 80 per cent or less on the test"; "Half of the class scored 80 per cent or more on the test"; "The most frequent score on the test was 80 per cent.") All of these statements are based on the same set of data but each may give a different impression of how well the class did on the test.

The pupils usually enjoy collecting and bringing to class graphs that may give false impressions, and by learning how to recognize poorly made graphs they learn how to make graphs properly.

This unit also affords the opportunity to reinforce the study of other subjects. For example, in giving the pupils data with which to prepare graphs, the data may be selected from almost any other field such as economics, geography, physical education, science, and social studies. In presenting the data to the pupils, the data itself may be discussed. Also, the pupils may collect the data from other courses and bring it in for use in mathematics class. Cross-subjects activities of this nature are educationally rewarding.

Statistics tell us what has happened, what is happening, and aids us in predicting what is most likely to happen.

The pupils may be asked to list various examples of data-collecting and of activities based on, or greatly influenced by, data and statistics. A few such examples are determining insurance policy premiums, government budgets, television program ratings, sports data of all types, the stock market, and pupil examination grades, averages, and final marks.

A discussion of data and statistics probably would not be complete without mentioning the role of the electronic computer. The two greatest values of a computer are that it has a huge memory capability and that it can do lightning-fast computations. A computer can be fed great quantities of data upon which it can quickly perform various operations to give the desired information. A computer is not "intelligent" in the usual sense of the word, but it is very, very fast. It can perform computations millions of times faster than the human brain. This is its greatest value in the field of data and statistics.

Often, short cuts are necessary in collecting data. In determining how many people are watching certain television programs on a given evening, it is a physical and economic impossibility to contact 180 million people in the United States to find out how many are watching each program.

In attempting to determine how many people are watching certain television programs or in attempting to predict the outcome of an upcoming election, the process of sampling is used.

Concept: In statistics, a sample is a subgroup selected from a larger group that is under consideration. The object in selecting the sample is to make the selection in such a manner that it is representative of the larger group.

That is, the results obtained by collecting data from the sample will give the same information as if the data had been collected from the entire group. Such a sampling is called a representative sampling. Selecting a sample that is representative can be quite difficult. In selecting a sample to predict the outcome of an election, different results would be obtained if the sample consisted of predominately Republicans or predominately Democrats. The sample would not be representative if it consisted of predominately men and one of the candidates has a strong appeal to women.

There is always uncertainty about the data obtained from samples because of the uncertainty as to whether or not the sample is representative of the entire group.

Many factors must be taken into account in selecting a sample. How, when, and where the sampling is done and the size of the sample all affect the results of the sampling. Selecting a sample and performing a sampling so that it will be representative can be quite a difficult task.

Tabulation of Data

Data is tabulated so as to put it into a more usable form.

Many types of data may be easily tabulated by the process of tallying. The ages of pupils in a mathematics class may be tabulated in this way. Suppose the ages of the pupils are: 15, 15, 14, 15, 13, 13, 13, 14, 15, 12, 15, 14, 15, 15, 14, 14, 13, 13, 16, 15, 15, 14, 14, 15, 14, 13, 14, 14. This may be tabulated as follows:

Age	Tally marks	Number
12	1	1
13	𝍸 1	6
14	𝍸 𝍸	10
15	𝍸 𝍸	10
16	1	1

There are several ways of *tabulating data*. The previous illustration showed how to list in some type of order all the different numbers that occur in the data and then show how often each score occurred by the use of *tally* marks.

Concept: The number of times each number appears is called the frequency of that number.

This type of tabulation of data is called a *frequency distribution*. *Cumulative frequency* shows, for any given score, the total frequency of all scores up to and including that score.

Given the following grades received by pupils on a certain examination: 85, 80, 85, 80, 84, 79, 82, 77, 81, 76, 81, 76, 80, 77. Make a tabluation showing the tally, frequency and cummulative frequency.

Answer:

Score	Tally Marks	Frequency	Cummulative Frequency
85	11	2	2
84	1	1	3
83		0	3
82	1	1	4
81	11	2	6
80	111	3	9
79	1	1	10
78		0	10
77	11	2	12
76	11	2	14
		14	

Sometimes the data is a set of numbers that differ so greatly that it is necessary to collect similar scores into groups. For example, the weights of 20 male adults in a certain group may be as follows:

158	162	148	167	169
144	151	152	155	157
166	180	177	168	156
171	173	183	159	165

Score	Tally Marks	Frequency	Cummulative Frequency
65-73	1111	4	23
0-65	11	2	25
		25	

For each of the following, make a tabulation of tally marks, frequency and cummulative frequency, first deciding whether or not grouping should be used. If grouping is used, group into five groups.

(a) The number of children in the families of the pupils in an eighth grade are: 1, 4, 1, 3, 2, 6, 1, 2, 2. 3, 2, 4, 2, 1, 1, 2, 3, 3, 4, 5, 2, 1, 4, 2, 2, 1.

(b) The number of telephone calls made from each of 30 different telephones in a day is:

5	18	6	3	0	12
14	19	5	7	3	2
0	1	9	14	6	11
8	3	7	7	9	5
4	0	2	13	14	5

(c) The average gasoline mileage of each of 25 new cars, in miles per gallon of gasoline, is:

17.4	21.6	13.1	14.0	13.6
21.0	18.4	15.2	13.8	14.6
15.5	16.7	16.8	15.9	13.9
14.2	15.7	17.3	18.8	19.1
24.0	27.0	26.3	25.8	24.4

(d) The height of each of the boys in a ninth grade gym class, to the nearest inch, is:

68	69	56	63	69	69	56
63	62	58	60	63	66	59
64	66	61	52	58	57	61
62	64	68	60	59	50	

Answers:

(a)

Children in family	Tally marks	Frequency	Cummulative Frequency
6	1	1	1
5	1	1	2
4	1111	4	6
3	1111	4	10
2	~~1111~~ 1111	9	19
1	~~1111~~ 11	7	26
		26	

No two men have the same weight. A frequency distribution that did not group the data would give no further information than the raw data itself. When it is obvious that the data should be grouped, the first problem is to decide intc how many groups the data should be divided. This decision should be based on the range of the data, the number of items forming the data, and the purpose for which the data is to be used. There are no hard and fast rules to use to determine the number of groups, but 10 to 15 groups are very often convenient. The data above ranges from 144 to 183 or a range of about 40 pounds. The data could be divided into 10 groups with an interval of 4 pounds each. The boundaries of the intervals should be defined so that there is no doubt as to where an item belongs. Below is a frequency distribution of the above data divided into 10 groups.

Weight Interval	Tally Marks	Frequency	Cummulative Frequency
180-183	11	2	2
176-179	1	1	3
172-175	1	1	4
168-171	111	3	7
164-167	111	3	10
160-163	1	1	11
156-159	1111	4	15
152-155	11	2	17
148-151	11	2	19
144-147	1	1	20
		20	

If the data had been grouped into 5 groups, the summary would be as follows:

Weight Interval	Tally Marks	Frequency	Cummulative Frequency
176-183	111	3	3
168-175	1111	4	7
160-167	1111	4	11
152-159	~~1111~~ 1	6	17
144-151	111	3	20
		20	

The fewer the number of groups that are used, the greater the information that is lost.

As another example, the following are the grades received by pupils on a mathematics test: 98, 88, 93, 84, 76, 77, 84, 92, 65, 73, 88, 92, 96, 100, 87, 86, 42, 58, 69, 77, 76, 80, 90, 80, 66. This data may be tabulated in groups as follows:

Score	Tally Marks	Frequency	Cummulative Frequency
92-100	~~1111~~ 1	6	6
83-91	~~1111~~ 11	7	13
74-82	~~1111~~ 1	6	19

(b)	Number of calls	Tally marks	Frequency	Cummulative Frequency
	16 - 19	11	2	2
	12 - 15	~~1111~~	5	7
	8 - 11	1111	4	11
	4 - 7	~~1111~~ ~~1111~~	10	21
	0 - 3	~~1111~~ 1111	9	30
			30	

(c)	Miles per gallon	Tally marks	Frequency	Cummulative Frequency
	25.1-28.0	111	3	3
	22.1-25.0	11	2	5
	19.1-22.0	111	3	8
	16.1-19.0	~~1111~~ 1	6	14
	13.1-16.0	~~1111~~ ~~1111~~ 1	11	25
			25	

(d)	Height in inches	Tally marks	Frequency	Cummulative Frequency
	66 - 69	~~1111~~ 1	6	6
	62 - 65	~~1111~~ 11	7	13
	58 - 61	~~1111~~ 1111	9	22
	54 - 57	111	3	25
	50 - 53	11	2	27
			27	

Representation of Cata

The newspapers and various reports contain examples of the most common types of graphs used as a pictorial representation of data: *bar graphs*, *broken line graphs*, and *circle graphs*. Almost any type of data can be represented by more than one type of graph, but each type of graph has its advantages in representing specific types of data.

Concept: The bar graph is often used to represent isolated number facts.

For instance, a bar graph may be used to represent the heights of the 10 highest mountains in North America or the length of the world's longest rivers. Such data is composed of number facts assigned to relatively unchanging items.

Concept: When the data is composed of number facts assigned to changing items and the purpose of the graph is to show the direction of change, the broken line graph is very useful.

Broken line graphs can be used to show trends, cycles, change in a variable or relationships between variables.

Concept: Circle graphs are useful for representing the relationship between portions of a whole and the whole.

For instance, a circle graph is useful for representing the items that make up a family budget. The data represented by circle graphs is often given as a ratio of a part to the whole expressed as a per cent. Such data may also be represented by a rectangular distribution graph which employs the use of a rectangle instead of a circle. In circle graphs and in rectangular distribution graphs, the area of each region of each graph is drawn proportional to the data.

If a circle graph is used to represent a family budget, and 20 per cent of the budget is for rent, then 20 per cent of the circle must be used to represent the rent portion of the budget. Here again, what is meant more precisely is that 20 per cent of the circular region must be used for this purpose. To do this, central angles are drawn at the center of the circle. The sum of all such angles will be 360°. The angle used to form the portion of the circle representing rent will be 20 per cent of 360° or 72°. An angle of 72° is drawn at the center of the circle intersecting the circle to form a sector which is 20 per cent of the entire circular region.

In making bar graphs, the bars may be drawn either horizontally or vertically. The width of the bars is the same and the width of the spaces between bars is the same.

One problem with both broken line graphs and bar graphs is that of determining what scales to use. One method of determining the scale is as follows. Determine what is the largest number that has to be represented on the graph. Determine how many graph paper units are available to represent this number. Divide the largest number that must be represented on the graph by the number available graph paper units. Each unit on the graph paper should represent the nearest larger convenient number. For instance, if the number 148,842 is the greatest number that must be represented on the graph, and if there are 15 graph paper units available, then dividing 148,842 by 15 gives a quotient of about 9,900. Each unit on the graph paper should then represent 10,000 for this scale.

Frequency distributions of grouped data may be represented by a histogram.

Concept: A histogram is somewhat like a vertical bar graph, except that there are no spaces between the bars.

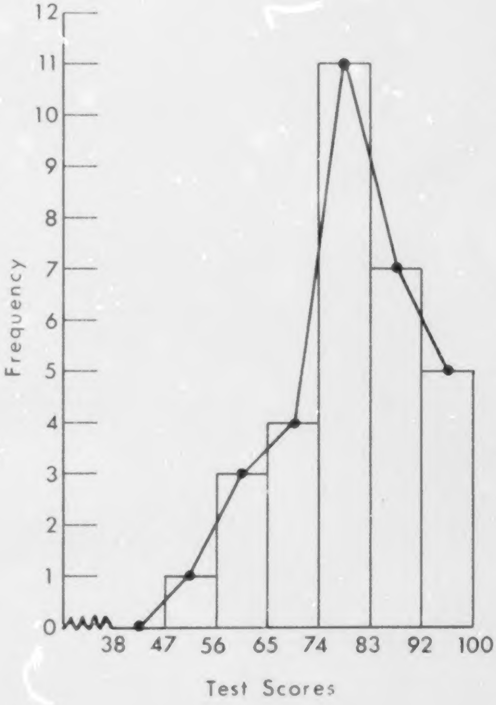
The intervals are marked off as the horizontal axis and the frequency is shown on the vertical axis.

A convenient interval length is one which will yield a maximum of ten to twenty intervals, though in the initial work six to ten intervals are an ample number with which to have students work. Adjacent intervals have a common endpoint. End points should be chosen with care in order that data fall within the intervals; at times this is impossible and by convention any data falling on an endpoint will be assigned to the lower interval.

Scores on Test	Tally Marks	Frequency
92 - 100	~~1111~~	5
83 - 91	~~1111~~ 11	7
74 - 82	~~1111~~ ~~1111~~ 1	11
65 - 73	1111	4
56 - 64	111	3
47 - 55	1	1

There are no scores between 0 and the 47 - 55 interval. This portion of the graph may be omitted and this is indicated by the zig-zag portion on the horizontal scale. The use of graphs in which a portion of the graph has been omitted can lead to false impressions of the data being represented. This is discussed in a later section.

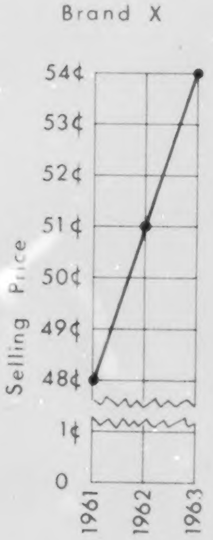
Joining the midpoints of the upper bases of a histogram results in a broken line graph called a *frequency polygon.* The midpoints themselves represent the meaningful data. The line segments connecting the midpoints are used simply to show trends in the variation of the data from one midpoint to the next midpoint. The points on the line segments connecting midpoints do not represent meaningful data.



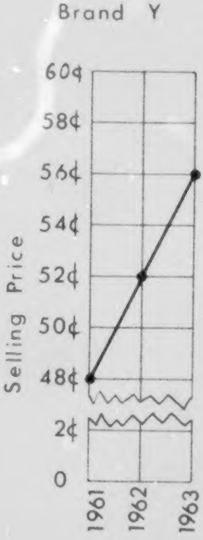
Relative frequency is the percent of the total number of scores that a frequency represents. Decimal numerals are used on a vertical line at the right of a histogram or frequency polygon, to illustrate relative frequency.

Interpretation of Graphs

Care must be exercised when interpreting graphs. A common saying can be slightly altered to say that graphs do not lie but liars use graphs. The meaning behind this saying can be demonstrated by the two graphs below. Graph A shows the price of Brand X on the first of each of three consecutive years. Graph B shows the price of Brand Y on these same dates. At first glance, it might appear that the percentage increase in Brand X has been far greater than the percentage increase in Brand Y during the same period.



A



B

A closer examination of the graphs show that Brand X increased from $0.48 to $0.54 while Brand Y increased from $0.48 to $0.56. Brand Y actually had an increase in price $0.02 greater than Brand X.

How can graphs be used to create such false impressions? An examination of the graphs shows that the two graphs do not have the same vertical scale. Graph A has a vertical scale in which one unit represents one cent. Graph B has a vertical scale in which one unit represents two cents. The zig-zag portion of the vertical scales shows that a portion of each graph has been omitted. In each graph the portion that has been omitted is different from the portion that has been omitted in the other graph. This makes the difference in vertical scales less obvious. Omitting a portion of a graph is not necessarily an indication that an attempt is being made made to create a false impression. Omitting a portion of a graph is very useful when the purpose of the graph is to show a trend or cycle in a variable and the variation percentage-wise is quite small.

Pupils should be asked to look for illustrations in which statistics are used to show only what is desired, i.e., "4 out of 5 dentists recommend --", which might mean only five were asked or only a certain 5 are being considered. The inference is being suggested however that of all the dentists in the country, 80% would recommend Brand Q.

Measures of Central Tendency

Central tendency refers to the part of the distribution at which many of the scores are concentrated.

Concept: The mode is the number which occurs most frequently in a set of data. A set of data may have more than one mode.

Concept: The mean is the arithmetic average of a set of data and is obtained by finding the sum of all the numbers in the set and dividing by the cardinal number of the set.

Concept: The median is the middle number in a set of data which has been arranged in order of size.

If the set of data contains an odd number of elements, the median is the value of the middle number. Thus, if there are n elements in the set, where n is an *odd* number, and these are arranged in order of size, the median occupies position $\frac{n+1}{2}$

If the set of data contains an even number of elements the median may be found as follows: when n is even and the numbers are arranged in order of size, the median is the arithmetic mean of the numbers in the positions, $\frac{n}{2}$ and $\frac{n}{2} + 1$.

Let us see how an analysis of data may be made.

The following is a set of salaries of 12 employees:

$5,200
$5,700
$5,500
$5,800
$5,400
$16,400
$5,400
$5,200
$5,600
$5,600
$5,500
$5,200

Arranged in increasing order the salaries are: $5,200, $5,200, $5,200, $5,400, $5,400, $5,500, $5,500, $5,600, $5,600, $5,700, $5,800, $16,400. The most frequent salary is $5,200. Therefore, the mode is $5,200. The middle salary is $5,500. Therefore, the median is $5,500. The arithmetic average or mean is $6,375. Notice how much higher the mean is than either the mode or median.

Each of the following is a true statement, based on this data:

- The most common salary among the employees is $5,200.
- Half of the employees receive a salary of $5,500 or more.
- Half of the employees receive a salary of $5,500 or less.
- The average salary of the employees is $6,375.

Each of these statements is a summary of the same data, yet the impression given by each of these statements can be entirely different. In this example the median gives the most accurate average as an indication of central tendency. The reason that the mean is so high is due to the $16,400 salary in the data. If this salary is changed to $5,700, the mode remains unchanged, the median remains unchanged, but the mean drops from $6,375 to $5,483. This demonstrates the effect that one number can have on the mean of a set of data.

The mode of data that has been grouped in a frequency distribution would be the interval with the greatest frequency. After the data has been grouped there is no way of determining the most frequent number in the original set of data.

The median of data that has been grouped cannot be precisely determined but the following is a description of how it can be approximated.

Interval Yearly Salary	Frequency
$9,000 - $10,000	1
$8,000 - $8,999	4
$7,000 - $7,999	6
$6,000 - $6,999	15
$5,000 - $5,999	40
$4,000 - $4,999	35
	N = 101

There are 101 salaries. The 51st salary will be the median salary. The 51st salary will be in the $5,000 - $5,999 interval. It can, therefore, be stated that the median salary is in the range $5,000 - $5,999. Counting down from the top, the 51st salary would be the 25th salary in the $5,000 - $5,999 group. Counting from the bottom it would be the 16th salary in that group. An approximation of the median salary would be $\frac{24.5}{40}$ of the way from $5,999 - $5,000 or $\frac{15.5}{40}$ of the way from $5,000 to $5,999. This would give an approximate median of $5,387. This is only an approximation because the calculation is based on the assumption that the salaries are evenly distributed within the group, which probably is not true.

An approximation of the mean of a grouped frequency distribution may be obtained by assuming that each number in each group has the value of the midpoint of that group. The mean is then calculated in the usual manner. A shortcut would be to multiply the midpoint of each group by the frequency of that group, add all these products, and then divide the sum by the total frequency. This is an approximation of the mean because it is based on the assumption that all the data are evenly distributed about the midpoint in each group.

Concept: The mean, median, and mode are numbers used as values typical or representative of a set of data.

Students and parents are often confused by percentile "scores" given as results of standardized tests. The explanation which follows should help to clarify the meaning of *percentiles*.

One method of indicating what portion of the pupils are below a certain pupil is by the use of percentile rank. If 4,000 pupils take an examination and one pupil is ranked 1,200, then 2,800 or 70 per cent of the pupils taking the test are below him. His percentile rank is given as 70. This means that 70 per cent of the pupils are below him. Percentile ranks are used a great deal in scoring standardized examinations, the purpose of which is not "pass" or "fail" pupils but to indicate ability compared to all the other pupils.

Data and Data Collection

Data for statistical analysis can not be collected in a blind and unthinking manner. There are pitfalls to be avoided merely in the collection of data, before any mathematical operations are performed on this data.

The process of collecting data can be very expensive. For example, television broadcasting networks wish to know how many people are watching their programs. If 25 million people are watching television in the United States on a certain Tuesday evening, it would cost a network a huge sum of money to contact all 25 million viewers and inquire as to which program they were watching. The task of contacting 25 million people in one evening is a practical impossibility. Instead of contacting all 25 million viewers, only a small sample, perhaps 2,000 viewers, are contacted and asked what program they are watching. This is called sampling. Firms that specialize in this activity are called television rating services.

If a television rating service contacts 2,000 people in different cities in various sections of the country, and if 27 per cent of the people contacted are watching a specific television program, does this mean that 27 per cent of all television viewers in the United States are watching that particular program?

Answer: No, not necessarily

If a sampling is made in such a manner that the results are the same as would be obtained if each person to whom the question pertains was contacted, the sampling is called a representative sampling. There are many factors that will determine whether or not a sampling will be representative. Such factors as how the sampling was taken, when it was taken, where it was taken, and the size of the sampling are a few such factors.

Each of the following is an example of a sampling which is not representative. The results of the sampling are different from the results that would be obtained if each person to whom the question pertains was contacted. For each sampling, indicate why it is not a representative sampling.

(a) In attempting to predict the outcome of an upcoming presidential election, 20,000 high school pupils in 100 high schools across the country were asked which candidate they wished to see elected.

(b) To determine how many people in the United States watched a certain television program each week, 42 people in seven different cities were interviewed.

Answers:

(a) The question was asked of the wrong people. High school students are not old enough to vote and will not be voting in the election. The question should have been asked of people who actually will be voting in the election.

(b) The sampling is too small. The number of people questioned is too small and the number of cities involved in the sampling is too small to be representative.

XII. PROBABILITY

The material in this unit is suggested as an optional topic in the 8th grade. Probability should be approached on the junior high level on a strictly intuitive basis.

A discussion of such words as: probably, might, chance, is a good starting point. Just what is meant when the following statements are made?

(a) It will probably rain to-night.
(b) I might get a new coat.
(c) There is a chance my father will take my fishing on Saturday.
(d) It might snow in July in New York City.
(e) He has a chance to go to the center of the earth tomorrow.

There should be an overall discussion regarding the statements and an emphasis on the likelihood of the events mentioned really occuring. The likelihood of the first three events happening is forseeable; while the likelihood of the last two is, for all practical purposes, nonexistent.

Experiments involving the tossing of coins form an effective laboratory approach to probability. Some of the fundamental concepts of probability can be illustrated on an informal, intuitive basis in such coin-tossing experiments.

When a coin is tossed in the air the probability of its landing heads up is the same as the probability of its landing tails up. In discussing the probability of one side of a tossed coin facing upward when the coin lands, it is assumed that the coin will not land on its edge. Therefore, when a coin is tossed there are only two possible *outcomes*, either the coin will land heads up or it will land tails up. In determining the probability of a tossed coin turning up heads, heads may be called the favorable *outcome*. Then, there are two possible outcomes, one of which is favorable.

Concept: The measure of the probability of a specific outcome occurring is the ratio of the number of favorable outcomes to the number of possible outcomes. The formula is

$$\text{Probability} = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}}$$

The probability of heads is therefore, $P = \frac{1}{2}$. In determining the probability of the outcome being tails, tails is taken as the favorable outcome. There are two possible outcomes so the measure of the probability of tails is $P = \frac{1}{2}$.

When a coin is tossed twice, there are four possible outcomes: (H, H); (H, T); (T, H); and (T, T). *Notice that the outcome of first heads and then tails is not the same outcome as the first tails and then heads.* These are two different outcomes.

Instead of tossing one coin twice, we may toss two coins once and get the same probabilities. If coins of different denomination are used, it is easier for students to see that (H, T) is not the same outcome as (T, H).

The probability of each of the four outcomes is as follows:

P (H, H) = $\frac{1}{4}$

P (H, T) = $\frac{1}{4}$

P (T, H) = $\frac{1}{4}$

P (T, T) = $\frac{1}{4}$

When a coin is tossed twice the probability of getting one head and one tail, in either order, is $P = \frac{2}{4}$ or $P = \frac{1}{2}$. In this instance, both (H, T) and (T, H) are favorable outcomes. There are four possible outcomes, two of which are favorable. The ratio of favorable outcomes to total possible outcomes is $\frac{2}{4}$ or $\frac{1}{2}$.

When a coin is tossed in the air three times, there are eight outcomes:

(H, H, H); (H, H, T); (H, T, H); (H, T, T);
(T, H, H); (T, H, T); (T, T, H); (T, T, T).

The probability of three heads is $\frac{1}{8}$.

The probability of three tails is $\frac{1}{8}$.

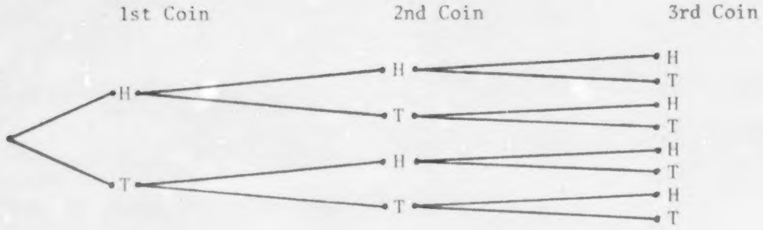
The probability of two heads and one tail is $\frac{3}{8}$.

The probability of one head and two tails is $\frac{3}{8}$.

The probability of all the coins turning up the same is $\frac{1}{4}$.

The probability of only two coins turning up the same is $\frac{3}{4}$.

A *tree diagram* may be used to show the possible outcomes. Thus for 3 consecutive coin tosses or one toss of three different coins, we may make the following diagram.



We read the paths from left to right to list the outcomes. There are 2 ways one coin may fall, (H), (T). There are 4 ways 2 coins may fall, (H, H), (H, T), (T, H), (T, T). There are eight ways three coins may fall, (H,H,H), (H,H,T), (H,T,H), (H,T,T), (T,H,H), (T,H,T), (T,T,H), (T,T,T).

Concept: If an event has only two possible outcomes and that event is repeated n number of times, the number of possible sequences or individual outcomes is 2^n.

If a coin is tossed four times there are 2^4 or 16 possible sequences or individual outcomes. If a coin is tossed five times, there are 2^5 or 32 possible sequences or individual outcomes.

Concept: To determine the measure of success of a specific outcome, the only information needed is the total number of possible outcomes and the number of these outcomes that are favorable. If all the possible outcomes are favorable, then the probabili- of the favorable outcome occurring is $\frac{1}{1}$ *or 1. If none of the outcomes is favorable, the probability of a favorable event is zero. Therefore, the range of the measures of the probabilities of events is from zero to one.* $0 \le P \le 1$.

Concept: The probability of success plus the probability of failure is always equal to 1.

Another popular method of studying the theory of probability is by the use of dice. It is advisable to restrict the discussion at first to the use of two dice of different color. The reason for this is that the outcome of a 3 on the first die and a 2 on the second die is not the same outcome as a 2 on the first die and a 3 on the second die. These are two of the 36 different possible outcomes. The use of dice colored differently helps to prevent difficulties in understanding this concept.

In considering the outcomes of rolling two differently colored dice, one red and one white, the outcomes may be listed as ordered number pairs. The use of ordered-number pairs will help to stress the point that if the first number of the ordered number pair is the number on the red die and the second number is the number on the white die, the ordered number pairs (3, 2) and (2, 3) are different ordered number pairs and are therefore different possible outcomes. When two differently colored dice are rolled, there are 36 possible outcomes. The probability of an outcome occurring is the ratio of the number of favorable outcomes to the number of total outcomes.

Concept: A sample space, is a listing of possible outcomes.

If the outcomes are taken to be the *sums* of the faces uppermost when two differently colored dice are rolled, the sample space is shown in the following diagram.

Red

White

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Total of 36 outcomes

Use the sample space in the preceding diagram to find answers to the following questions:

(a) Find the probability of a total of seven showing
(b) Find the probability of a total < two showing
(c) Find the probability of a total of three or six showing

Answer:

(a) Reading the sample space we see that there are six possible ways to get a seven out of the total of the thirty-six possible outcomes. Hence the probability is $\frac{6}{36} = \frac{1}{6}$

(b) Again the sample space is checked and there are no possible outcomes which are <2, so the probability is $\frac{0}{36} = 0$.

(c) Checking the sample space; there are two outcomes of three and five outcomes of six. The total possible outcomes are thirty six, so the probability is: $\frac{2+5}{36} = \frac{7}{36}$

If question (c) were reworded to read "... a total of three *and* six," the probability would be zero since both outcomes are not possible at the same time.

In the following exercises use the formula

$$\text{Probability} = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}}$$

Using a single cubical die numbered from 1 through 6, find the probability of the following events taking place if the die is thrown?

a) The number showing is one?
b) The number showing is five?
c) The number showing is < 4?

Answers: (a) $\frac{1}{6}$ (b) $\frac{1}{6}$ (c) $\frac{3}{6} = \frac{1}{2}$

Using a standard 52 card deck of cards find the probability of the following cards being drawn. (After each draw the card is replaced.)

a) The ace of hearts
b) Any king
c) The queen of diamonds or clubs
d) Any face (picture) card

Answers: (a) $\frac{1}{52}$ (b) $\frac{4}{52} = \frac{1}{13}$ (c) $\frac{2}{52} = \frac{1}{26}$ (d) $\frac{12}{52} = \frac{3}{13}$

One serious common misconception of probability pertains to the probability of *independent events*. When a coin is tossed six times and heads comes up each time, many people think that the probability of heads coming up a seventh time is very small. When the coin is tossed a seventh time the outcomes of the previous tosses have no effect on this toss. Each toss is independent of the previous toss. When the coin is tossed a seventh time, the probability of heads on that one toss is still $\frac{1}{2}$. Of course, when a coin is tossed seven times, the total number of possible outcomes is 2^7 or 128. The probability of getting seven heads in seven tosses is $\frac{1}{128}$. However, after the sixth toss has taken place, the probability of heads on the seventh toss is $\frac{1}{2}$.

This leads to a discussion of the probability of either of two events occurring and the probability of both of two events occurring.

Concept: If two events are mutually exclusive, that is, if it is impossible for both to occur at the same time, the probability of one event or the other event occurring is the sum of probability of the first event and the probability of the second event. P (A or B) = P (A) + P (B)

When a coin is tossed, the probability of either heads or tails turning up is $\frac{1}{2} + \frac{1}{2}$ or 1. It is a sure thing. When a coin is tossed, the outcome will be either heads or tails.

When one die is tossed the probability of a 2 or a 6 turning up is P (2 or 6) = $\frac{1}{6} + \frac{1}{6}$ or $\frac{1}{3}$. This also applies to more than two events if all events are mutually exclusive.
P (A or B or C) = P (A) + P (B) + P (C).

Concept: In independent events, the probability of both of two events occurring is the product of the probability of the first and the probability of the second event. P (A and B) = P (A)·P (B).

When a coin is tossed twice, the probability of heads both times is $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$ or $\frac{1}{4}$. When the coin is tossed three times, probability of tails all three times is $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$ or $\frac{1}{8}$.

When a pupil is taking a multiple choice test in which there are five choices for answers to each question, and he has no idea as to what the answer to a question is, the probability of his getting it correct is $\frac{1}{5}$. If there are two questions to which he does not know the answer, the probability of his getting both questions correct is $\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)$ or $\frac{1}{25}$. If there are five questions to which he does not know the answer, the probability of his getting all five correct is $\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)$ or $\frac{1}{3125}$. If there are ten questions to which he does not know the answer, the probability of his getting all ten questions correct is
$\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)$ or $\frac{1}{9,765,625}$.

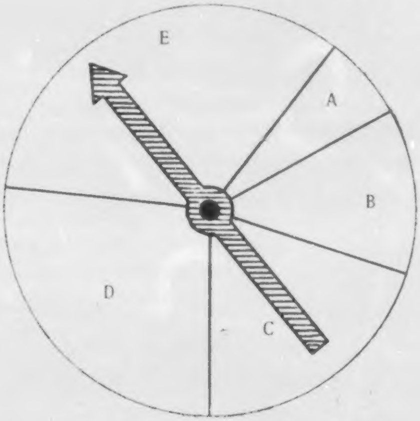
Simple exercises such as the following are useful.

A drawer contains 5 white socks and 10 black socks. What is the probability that on the first draw a white sock will be picked? What is the probability that on the first draw a black sock will be picked?

Answer: White, $\frac{5}{15} = \frac{1}{3}$ Black, $\frac{10}{15} = \frac{2}{3}$

A woman has a half dollar, three quarters, two dimes, and five pennies in her wallet. What is the probability that she will draw out a quarter on her first try?

Answer: $\frac{3}{11}$



The above diagram represents a "spinner" with the dial divided into five sectors as follows: A = 24°, B = 48°, C = 72°, D = 96° and E = 120°. When the pointer is spun, what is the probability of it stopping on: (a) an A, (b) a D, (c) a C or an E, (d) an A, B or C, (e) the letters B, A, D in three successive spins?

Answers:

(a) $\frac{24}{360} = \frac{1}{15}$, (b) $\frac{96}{360} = \frac{4}{15}$, (c) $\frac{1}{5} + \frac{1}{3} = \frac{8}{15}$,

(d) $\frac{1}{15} + \frac{2}{15} + \frac{1}{5} = \frac{2}{5}$, (e) $\left(\frac{2}{15}\right)\left(\frac{1}{15}\right)\left(\frac{4}{15}\right) = \frac{8}{3375}$

As enrichment material, the teacher may wish to suggest to students that they try to construct dials with sectors which would simulate the outcomes of tossing three coins or rolling two dice. The angles of the sectors should be such as to give the correct probabilities of the various outcomes.